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A state space approach via Discrete Hartley transform of type H^{I},H^{II}, H^{III}, and $H^{\mathbb{IV}}$ For the solution of the State model of Linear time-invariant systems (L.T.I.S's.)

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ABSTRACT

In this paper, we deal with linear time-invariant systems (L.T.I.S.), and single-input single-output systems (S.I.S.O.S.). We intend to identify two types of L.T.I.S's. in Hartley series, namely homogeneous and non-homogeneous system. The presented Hartley series domain identification pivots upon the samples of involved functions. We solve the problem of system identification where the systems are described by state space models, using the concept of Hartley transform of type H^{I},H^{II}, H^{III}, and $H^{\mathbb{IV}}$domain technique. This method is based on a new integration operational matrix, which relates Hartley transform of type $H^{\mathbb{I}}$,H^{II}, H^{III}, and $H^{\mathbb{IV}}$And their integration, is defined. Examples of homogeneous and non-homogeneous state-equation are used to illustrate the method. We show that our method is very simple as it is very fast and so computationally very attractive.

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1. INTRODUCTION

The problem of the state model of linear time-invariant systems is very interesting in the literature [1-5]. To approximate the solution of homogeneous and non-homogeneous state equations, they used the integration operational matrices method for non-sinusoidal functions to solve this problem. This method was found in piecewise constant functions such as Haar, Walsh, Block Pulse, general hybrid orthogonal functions, and others.

In this paper, we will give a new numerical approximation solution for the state model of L.T.I.S's via sinusoidal functions denoted by the Hartley transform. The Hartley transform, first introduced by the American electronic researcher Ralph V. L. in 1942 [6], is a real-valued alternative to the Fourier transform.

Consider the Hartley series representation, i.e. the representation in terms of sine-cosine functions of a square-integrable function $\mathcal{S}(t)$In [0,1) is given via :

$$\mathcal{S}(t) \cong \sum_{\ell=0}^{n-1} \zeta_\ell \mathcal{H}_\ell(t) = \boldsymbol{\zeta}^T \boldsymbol{\mathcal{H}}(\boldsymbol{t}) \quad (1)$$

Where,

$$\mathcal{H}_\ell(t) = cas(2\ell\pi t) = sin(2\ell\pi t) + cos(2\ell\pi t)$$

And

$$\zeta_\ell = \int_0^1 \mathcal{S}(t)\,\mathcal{H}_\ell(t)$$

The spectral vector $\boldsymbol{\zeta}$ is given via

$$\boldsymbol{\zeta} = [\zeta_0\ \zeta_1\ \cdots\ \zeta_{n-1}]^T$$

and

$$\boldsymbol{\mathcal{H}}(t) = [\mathcal{H}_{\mathbf{0}}(t) \;\; \mathcal{H}_1(t) \; \cdots \; \mathcal{H}_{n-1}(t)]^T.$$

Thus, for each $0 \leq i$,j≤n-1, the fourth type of Hartley transform can be defined by, [7-18]:

- The Hartley transform of type $H^{\mathbb{I}}$ **:** $[\boldsymbol{\mathcal{H}}_n]_{ij} = \frac{1}{\sqrt{n}} cas(\frac{2ij\pi}{n})$
- The Hartley transform of type$H^{\mathbb{II}}$ **:** $[\boldsymbol{\mathcal{H}}_n]_{ij} = \frac{1}{\sqrt{n}} cas(\frac{i(2j+1)\pi}{n})$
- The Hartley transform of type$H^{\mathbb{III}}$ **:** $[\boldsymbol{\mathcal{H}}_n]_{ij} = \frac{1}{\sqrt{n}} cas(\frac{(2i+1)j\pi}{n})$
- The Hartley transform of type$H^{\mathbb{IV}}$ **:** $[\boldsymbol{\mathcal{H}}_n]_{ij} = \frac{1}{\sqrt{n}} cas(\frac{(2i+1)(2j+1)\pi}{2n})$

All types of H.T. are orthogonal matrices of order n.
The integration of $\boldsymbol{\mathcal{H}}(t)$ can be defined mathematically via:

$$\int_0^t \boldsymbol{\mathcal{H}}(\sigma)d\sigma \approx \mathfrak{P}\boldsymbol{\mathcal{H}}(t) \quad (2)$$

Where, $\mathfrak{P}$ the Hartley operational matrix of integration, which is a square matrix of dimension $n \times n$. Thus, we introduce a complete orthonormal system called the block-pulse system $\{\mathbb{B}_\gamma(t)\}_{\gamma=0}^{\infty}$, [5] as:

$$\mathbb{B}_\gamma(t) = \begin{Bmatrix} 1 & \alpha_1 \leq t < \alpha_2 \\ 0 & \text{o.w.} \end{Bmatrix} \quad (3)$$

Where, $\alpha_1 = \frac{d}{r}$,$\alpha_2 = \frac{d+1}{r}$, d=0,1,2, . . . ,r-1, $r = 2^s$ and sIs a positive integer.

Let $\mathbb{B}(t) = [B_0(t) \;\; B_1(t) \;\; B_2(t) \;\; . \; . \; . \;\; B_{n-1}(t)]^T$Be a block-pulse vector, then the I.O.M. of $\mathbb{B}(t)$Is square matrix obtained by integrating each element in $\mathbb{B}(t)$, And expressing mathematically by the following formula:

$$\int_0^t \mathbb{B}(\sigma)d\sigma = \mathcal{P}\,\mathbb{B}(t) \quad (4)$$

Where,

$$\mathcal{P} = \frac{1}{n}\begin{pmatrix} 0.5 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0.5 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0.5 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0.5 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0.5 \end{pmatrix} \quad (5)$$

Let us show the I.O.M. of $\boldsymbol{\mathcal{H}}(t)$ in terms of the block-pulse system:
First, because the $\mathcal{H}_\ell(t)$ are belong to square-integrable functions in [0,1), then they can be approximated by a n −set of block-pulse functions via:

$$\mathcal{H}_\ell(t) = \sum_{j=0}^{n-1} \mathcal{k}_{\ell j}\, \mathbb{B}_j(t) \quad (6)$$

Where, ℓ=0,1,2, . . .,n-1, and $\mathcal{k}_{\ell j}$ denotes the coefficients of $\mathcal{H}_\ell(t)$Which their values are the H^I,or H^{II}, or H^{III}, or $H^{\mathbb{IV}}$Matrices respectively. Equation(6) can be written in a square matrix form in order n:

$$\boldsymbol{\mathcal{H}}(t) = [\mathcal{k}]_n[\mathbb{B}]_n \quad (7)$$

By substituting equation (7) into equation (2), we have

$$\int_0^t [\mathcal{k}]_n[\mathbb{B}]_n d\sigma = \mathfrak{P}\,[\mathcal{k}]_n[\mathbb{B}]_n$$

From equation (4), we obtain

$$[\mathcal{k}]_n\mathcal{P}\,[\mathbb{B}]_n = \mathfrak{P}\,[\mathcal{k}]_n[\mathbb{B}]_n$$

Therefore, we get the I.O.M. of $\boldsymbol{\mathcal{H}}(t)$In terms of the block-pulse system as follows:

$$\mathfrak{P} = [\mathcal{k}]_n\mathcal{P}\,([\mathcal{k}]_n)^{-1} \quad (8)$$

2. Methodology (STATE-SPACE (S.S.) APPROACH TO H^I,H^{II}, H^{III}, and $H^{\mathbb{IV}}$)

S.S. analysis is a modern approach to the design and analysis of control systems. Conventional methods for the design and analysis of control systems are based on the transfer function method (T.F.M.). The T.F.M. has many drawbacks, such as the transfer function being defined under zero initial conditions, applicable to linear time-invariant systems (L.T.I.S), and single-input single-output systems (S.I.S.O.S), and does not provide information regarding the internal state of the system.

2.1. STATE MODEL OF LINEAR TIME-INVARIANT SYSTEMS (L.T.I.S's.)

Definition (2.1.1): The state of a dynamic system is the smallest set of variables called state variables (S.V.'s). S.V.'s satisfy that the knowledge of these variables at the initial condition time $t = t_0$ with the knowledge of input for $t \geq t_0$, completely determines the behavior of the system at any time $t \geq t_0$.
Definition (2.1.2): State variables are a set of variables that describe the system at any instant.
Definition (2.1.3): "State vector '': A vector whose elements are the S.V.'s.
Definition (2.1.4): The $\mathbb{m}$ −dimensional space whose co-ordinate axes consist of the $x_1$axis, x_2 axis,⋯ , $x_{\mathbb{m}}$ axis, (where x_1 , x_2,⋯,$x_{\mathbb{m}}$ are state variables:) is called a S.S.

The state model of a system consists of a state equation and an output equation. The state equation of a system is a function of state variables and inputs. For L.T.I. multiple input multiple output systems (M.I.M.O.S's), the first derivatives of state variables can be expressed as a linear combination of state variables and inputs: mInputs: $w_1(t)$, $w_2(t)$,⋯,$w_m(t)$; pOutputs: $y_1(t)$, $y_2(t)$,⋯,$y_p(t)$; nState variables : $x_1(t)$, $x_2(t)$,⋯,$x_n(t)$. The different variables may be represented by the vectors as shown below
Input vector:

$$[\mathcal{W}(t)] = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_m(t) \end{bmatrix}; \text{Output vector: } [\mathcal{Y}(t)] = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}; \text{State variable vector } [\mathcal{X}(t)] = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

The S.V. representation can be arranged in the form of nNumber of first order differential equations as shown below:

$$\left.\begin{aligned} \dot{\mathcal{X}}_1 &= \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}w_1 + b_{12}w_2 + \cdots + b_{1m}w_m \\ \dot{\mathcal{X}}_2 &= \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}w_1 + b_{22}w_2 + \cdots + b_{2m}w_m \\ &\vdots \\ \dot{\mathcal{X}}_n &= \frac{dx_1}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}w_1 + b_{n2}w_2 + \cdots + b_{nm}w_m \end{aligned}\right\} \quad (9)$$

where the coefficients $a_{ij}$and $b_{ij}$are constants.
Equation (9) can be written in matrix form.

$$\begin{bmatrix} \dot{\mathcal{X}}_1 \\ \dot{\mathcal{X}}_2 \\ \vdots \\ \dot{\mathcal{X}}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \quad (10)$$

Or

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}\mathcal{W}(t) \quad (11)$$

Equation (11) is known as a state equation, $\mathcal{A}$ is state matrix of size (n×n), $\mathcal{B}$ is the input matrix of size (n ×m), $\mathcal{X}(t)$Is the state vector of size (n ×1), and $\mathcal{W}(t)$Is the input vector of size (m ×1).
The output at any time tAre functions of S.V.'s:

$$\mathcal{Y}(t) = \mathbb{F}(\mathcal{X}(t))$$

Thus, the output variables can be expressed as a linear combination of S.V.'s:

$$\left.\begin{array}{c} y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n \\ y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n \\ \vdots \\ y_p = c_{p1}x_1 + c_{p2}x_2 + \cdots + c_{pn}x_n \end{array}\right\} \quad (12)$$

where the coefficients c_{ij} are constants. Equation (12) can be written in matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (13)$$

Or in compact form

$$\boldsymbol{\mathcal{Y}}(\boldsymbol{t}) = \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{X}}(\boldsymbol{t}) \quad (14)$$

Equation (14) is called the output equation and $\boldsymbol{\mathcal{C}}$ is the output matrix of size ($p \times n$).
Let us consider the state equation of n -th order system is given via:

$$\begin{cases} \dot{\boldsymbol{\mathcal{X}}}(\boldsymbol{t}) = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{X}}(\boldsymbol{t}) + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(\boldsymbol{t}) \\ \boldsymbol{\mathcal{X}}(\boldsymbol{0}) = \boldsymbol{\mathcal{X}}_{\boldsymbol{0}}\text{: initial condition vector} \end{cases}$$

Taking Laplace transforms on both sides,

$$s\boldsymbol{\mathcal{X}}(s) - \boldsymbol{\mathcal{X}}(\boldsymbol{0}) = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{X}}(s) + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(s)$$
$$\boldsymbol{\mathcal{X}}(s)(sI - \boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{X}}(\boldsymbol{0}) + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(s)$$

Where, I is the identity matrix

$$\boldsymbol{\mathcal{X}}(s) = (sI - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{X}}(\boldsymbol{0}) + (sI - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(s) \quad (15)$$

Taking inverse Laplace transforms on both sides,

$$\boldsymbol{\mathcal{X}}(\boldsymbol{t}) = \boldsymbol{\mathcal{L}}^{-1}((sI - \boldsymbol{\mathcal{A}})^{-1})\boldsymbol{\mathcal{X}}(\boldsymbol{0}) + \boldsymbol{\mathcal{L}}^{-1}((sI - \boldsymbol{\mathcal{A}})^{-1})\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(s)$$

The solution of the state equation is,

$$\boldsymbol{\mathcal{X}}(\boldsymbol{t}) = \boldsymbol{\mathcal{Q}}(\boldsymbol{t})\boldsymbol{\mathcal{X}}(\boldsymbol{0}) + \boldsymbol{\mathcal{L}}^{-1}(\boldsymbol{\mathcal{Q}}(s)\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(s)) \quad (16)$$

Where, $\boldsymbol{\mathcal{Q}}(\boldsymbol{t}) = \boldsymbol{\mathcal{L}}^{-1}((sI - \boldsymbol{\mathcal{A}})^{-1})$(state transition matrix), and $\boldsymbol{\mathcal{Q}}(s) = (sI - \boldsymbol{\mathcal{A}})^{-1}$.

The output equation is:

$$\boldsymbol{\mathcal{Y}}(\boldsymbol{t}) = \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{X}}(\boldsymbol{t})$$

Taking Laplace transforms on both sides,

$$\boldsymbol{\mathcal{Y}}(s) = \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{X}}(s)$$

From equation (15), we get

$$\boldsymbol{\mathcal{Y}}(s) = \boldsymbol{\mathcal{C}}((sI - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{X}}(\boldsymbol{0}) + (sI - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(s))$$

$$\boldsymbol{\mathcal{Y}}(s) = \boldsymbol{\mathcal{C}}(sI - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{X}}(\boldsymbol{0}) + \boldsymbol{\mathcal{C}}(sI - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(s)$$

Setting $\boldsymbol{\mathcal{X}}(\boldsymbol{0}) = \boldsymbol{0}$, we have

$$\boldsymbol{\mathcal{Y}}(s) = \boldsymbol{\mathcal{C}}(sI - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(s) \quad (17)$$

The transfer function is

$$\frac{\mathcal{Y}(s)}{\mathcal{W}(s)} = \boldsymbol{\mathcal{C}}(s\mathrm{I} - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{B}} \quad (18)$$

2.2. SOLUTION OF STATE EQUATION IN TERMS OF H^{I},H^{II}, H^{III}, and $H^{\mathbb{IV}}$:

Let us consider the following state equation:

$$\begin{cases} \dot{\boldsymbol{\mathcal{X}}}(\boldsymbol{t}) = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{X}}(\boldsymbol{t}) + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{W}}(\boldsymbol{t}) \\ \boldsymbol{\mathcal{X}}(\mathbf{0}) = \boldsymbol{\mathcal{X}}_{\mathbf{0}}\text{: initial condition vector} \end{cases} \quad (19)$$

Suppose that $\boldsymbol{\mathcal{W}}(\boldsymbol{t})$ is square integrable in $t \in [0,1)$. It can be expressed as a Hartley series representation.

$$\boldsymbol{\mathcal{W}}(\boldsymbol{t}) = \boldsymbol{\zeta}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) \quad (20)$$

Thus, we also want to approximate $\dot{\boldsymbol{\mathcal{X}}}(\boldsymbol{t})$ into a Hartley series:

$$\dot{\boldsymbol{\mathcal{X}}}(\boldsymbol{t}) = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) \quad (21)$$

After integration, we obtain

$$\boldsymbol{\mathcal{X}}(\boldsymbol{t}) = \int_{\mathbf{0}}^{\boldsymbol{t}} \dot{\mathcal{X}}(\sigma)\, d\sigma + \boldsymbol{\mathcal{X}}_{\mathbf{0}} = \boldsymbol{\mathcal{T}} \int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{\mathcal{H}}(\sigma) d\sigma + \boldsymbol{\mathcal{X}}_{\mathbf{0}} = \boldsymbol{\mathcal{T}}\boldsymbol{\mathfrak{P}}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) + \boldsymbol{\mathcal{X}}_{\mathbf{0}} \quad (22)$$

Because $\boldsymbol{\mathfrak{P}}$, $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{\mathcal{X}}_{\mathbf{0}}$ are known, if we can solve for $\boldsymbol{\mathcal{T}}$, then we have $\boldsymbol{\mathcal{X}}(\boldsymbol{t})$. By substituting equations (20), (21), and (22) into equation (19), we get

$$\begin{gathered} \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{T}}\boldsymbol{\mathfrak{P}}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) + \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{X}}_{\mathbf{0}} + \boldsymbol{\mathcal{B}}\boldsymbol{\zeta}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) \\ \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) - \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{T}}\boldsymbol{\mathfrak{P}}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{X}}_{\mathbf{0}} + \boldsymbol{\mathcal{B}}\boldsymbol{\zeta}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) \\ \Rightarrow (\boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{T}}\boldsymbol{\mathfrak{P}}\boldsymbol{\mathcal{H}}(\boldsymbol{t}) = ([\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{X}}_{\mathbf{0}}\ 0\ 0\ \dots 0] + \boldsymbol{\mathcal{B}}\boldsymbol{\zeta})\boldsymbol{\mathcal{H}}(\boldsymbol{t}) \\ \Rightarrow \boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{T}}\boldsymbol{\mathfrak{P}} = [\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{X}}_{\mathbf{0}}\ 0\ 0\ \dots 0] + \boldsymbol{\mathcal{B}}\boldsymbol{\zeta} \equiv \mathbb{Q} \end{gathered} \quad (23)$$

Equation (23), can be written as

$$(\boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{T}}\boldsymbol{\mathfrak{P}} = (\boldsymbol{\mathcal{A}})^{-1}\mathbb{Q} \quad (24)$$

For nonsingular matrix $\boldsymbol{\mathcal{A}}$, eq. (24) is a Lyapunov matrix equation [18]. Therefore, we can get $\boldsymbol{\mathcal{T}}$ directly with $\boldsymbol{\mathcal{A}}$ and $\mathbb{Q}$ being given via applying lyap($(\boldsymbol{\mathcal{A}})^{-1}$,$-\boldsymbol{\mathfrak{P}}$, $(\boldsymbol{\mathcal{A}})^{-1}\mathbb{Q}$) from Matlab programmer. After $\boldsymbol{\mathcal{T}}$ is determined the solution $\dot{\boldsymbol{\mathcal{X}}}(\boldsymbol{t})$ is obtained. The solution $\boldsymbol{\mathcal{X}}(\boldsymbol{t})$ is easily found via substituting $\boldsymbol{\mathcal{T}}$ into eq. (22).

3. Results & discussion

In this section, we are using the I.O.M. of $\boldsymbol{\mathcal{H}}(t)$ for finding the numerical solution of state model of L.T.I.S.'s. All numerical solutions have been established by Matlab (R2021a) programming.

Example(3.1): Consider the set of homogenous differential equations to be given as

$$\begin{cases} \dot{\mathcal{X}}_1(t) = \mathcal{X}_2(t) \\ \dot{\mathcal{X}}_2(t) = -2\mathcal{X}_1(t) - 2\mathcal{X}_2(t) \\ \mathcal{X}_1(0) = 1 \\ \mathcal{X}_2(0) = -1 \end{cases}$$

It is required to find solutions for $\mathcal{X}_1(t)$ and $\mathcal{X}_2(t)$.

The approximated values of the $\mathcal{X}_1(t)$ and $\mathcal{X}_2(t)$ for the state equation for $n = 4$ derived by four types of Hartley transform i.e. $H^{\mathbb{I}}$, $H^{\mathbb{II}}$, $H^{\mathbb{III}}$, and $H^{\mathbb{IV}}$ are shown in Table 1. and Table 2.

Table 1. Approximated values of the $\mathcal{X}_1(t)$ for $n = 4$ in Hartley transform types.

Time t	Actual values $\mathcal{X}_1(t)$	H.T. of type $H^{\mathbb{I}}$	H.T. of type $H^{\mathbb{II}}$	H.T. of type $H^{\mathbb{III}}$	H.T. of type $H^{\mathbb{IV}}$
1/8	0.8772	0.8780	0.8780	0.8780	0.8407
3/8	0.6433	0.6448	0.6449	0.5943	0.5360
5/8	0.4388	0.4397	0.4398	0.3431	0.3613
7/8	0.2720	0.2718	0.2718	0.3087	0.4524

Table 2. Approximated values of the $\mathcal{X}_2(t)$ for $n = 4$ in Hartley transform types.

Time t	**Actual values** $\mathcal{X}_2(t)$	**H.T. of type** $H^{\mathbb{I}}$	**H.T. of type** $H^{\mathbb{II}}$	**H.T. of type** $H^{\mathbb{III}}$	**H.T. of type** $H^{\mathbb{IV}}$
1/8	-0.9737	-0.9756	-0.9756	-0.9756	-0.9681
3/8	-0.8852	-0.8900	-0.8899	-0.8798	-0.8562
5/8	-0.7453	-0.7509	-0.7509	-0.7154	-0.6932
7/8	-0.5879	-0.5929	-0.5929	-0.5596	-0.5786

Percentage errors for such approximation for $n = 4$ is presented in Table 3. and Table 4.

Table 3. Percentage errors for values approximation system state $\mathcal{X}_1(t)$ (n = 4) in Hartley transform types.

System state $\mathcal{X}_1(t)$ (n = 4)				
Time t	H.T. of type $H^{\mathbb{I}}$	H.T. of type $H^{\mathbb{II}}$	H.T. of type $H^{\mathbb{III}}$	H.T. of type $H^{\mathbb{IV}}$
1/8	0.0008	0.0008	0.0008	0.0365
3/8	0.0015	0.0016	0.049	0.1073
5/8	0.0009	0.001	0.0957	0.0775
7/8	0.0002	0.0002	0.0367	0.1804

Table 4. Percentage errors for values approximation system state $\mathcal{X}_2(t)$ (n = 4) in Hartley transform types.

System state $\mathcal{X}_2(t)$ (n = 4)				
Time t	H.T. of type $H^{\mathbb{I}}$	H.T. of type $H^{\mathbb{II}}$	H.T. of type $H^{\mathbb{III}}$	H.T. of type $H^{\mathbb{IV}}$
1/8	0.0016	0.0019	0.0019	0.0056
3/8	0.0048	0.0047	0.0054	0.0290
5/8	0.0056	0.0056	0.0299	0.0521
7/8	0.005	0.005	0.0283	0.0093

Example (3.2): Consider the set of non-homogenous differential equations to be given as

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}\mathcal{W}(t)$$

Where, $\mathcal{A} = \begin{bmatrix} -1 & -1/2 \\ 1 & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \mathcal{X}_1(0) \\ \mathcal{X}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\mathcal{W}(t)$ a unite step function.

The solution of the equation is $\mathcal{X}_1(t) = e^{-1/2}\sin(1/2t)$, and $\mathcal{X}_2(t) = -e^{-\frac{1}{2t}}(\cos(1/2t + \sin(1/2t)) + 1$.
The approximated values of the state equation, for $n = 8$ derived via four types of Hartley transform for example (3.2) are shown in Table 5. and Table 6.

Table 5. Approximated values of the $\mathcal{X}_1(t)$ for $n = 8$ in Hartley transform types.

Time t	Actual values $\mathcal{X}_1(t)$	H.T. of type $H^{\mathbb{I}}$	H.T. of type $H^{\mathbb{II}}$	H.T. of type $H^{\mathbb{III}}$	H.T. of type $H^{\mathbb{IV}}$
1/16	0.0303	0.0294	0.0294	0.0294	0.0345
3/16	0.0852	0.0844	0.0844	0.0934	0.1055
5/16	0.1331	0.1324	0.1324	0.1614	0.1735
7/16	0.1744	0.1738	0.1738	0.2202	0.2261
9/16	0.2095	0.2090	0.2090	0.2584	0.2532
11/16	0.2390	0.2386	0.2386	0.2677	0.2488
13/16	0.2632	0.2629	0.2629	0.2446	0.2119
15/16	0.2827	0.2825	0.2825	0.1906	0.1467

Table 6. Approximated values of the $\mathcal{X}_2(t)$ of example (3.2) for $n = 8$ in Hartley transform types.

Time t	Actual values $\mathcal{X}_2(t)$	H.T. of type $H^{\mathbb{I}}$	H.T. of type $H^{\mathbb{II}}$	H.T. of type $H^{\mathbb{III}}$	H.T. of type $H^{\mathbb{IV}}$
1/16	0.0010	0.0018	0.0018	0.0018	0.0022

3/16	0.0083	0.0089	0.0089	0.0095	0.0109
5/16	0.0220	0.0225	0.0225	0.0254	0.0283
7/16	0.0413	0.0416	0.0416	0.0493	0.0533
9/16	0.0653	0.0656	0.0656	0.0792	0.0833
11/16	0.0934	0.0935	0.0935	0.1121	0.1146
13/16	0.1248	0.1249	0.1249	0.1441	0.1434
15/16	0.1590	0.1589	0.1589	0.1713	0.1658

The comparative study of the error estimates for the example (3.2) for $n = 8$ via H.T. of types $H^{\mathbb{I}}$,H^{II}, H^{III} and $H^{\mathbb{IV}}$ domains are shown in Table 7. and Table 8.

Table 7. Percentage errors for values approximation system state $\mathcal{X}_1(t)$ (n = 8) in Hartley transform types.

System state $\boldsymbol{\mathcal{X}_1(t)}$ ($\boldsymbol{n}$ =8)				
Time $\boldsymbol{t}$	H.T. of type $H^{\mathbb{I}}$	H.T. of type $H^{\mathbb{II}}$	H.T. of type $H^{\mathbb{III}}$	H.T. of type $H^{\mathbb{IV}}$
1/16	0.0009	0.0009	0.0009	0.0042
3/16	0.0008	0.0008	0.0082	0.0203
5/16	0.0007	0.0007	0.0283	0.0404
7/16	0.0006	0.0006	0.0458	0.0517
9/16	0.0005	0.0005	0.0489	0.0437
11/16	0.0004	0.0004	0.0287	0.0098
13/16	0.0003	0.0003	0.0186	0.0513
15/16	0.0002	0.0002	0.0921	0.136

Table 8. Percentage errors for values approximation system state $\mathcal{X}_2(t)$ (n = 8) in Hartley transform types.

System state $\boldsymbol{\mathcal{X}_2(t)}$ ($\boldsymbol{n}$ = 8)				
Time $\boldsymbol{t}$	H.T. of type $H^{\mathbb{I}}$	H.T. of type $H^{\mathbb{II}}$	H.T. of type $H^{\mathbb{III}}$	H.T. of type $H^{\mathbb{IV}}$
1/16	0.0008	0.0008	0.0008	0.0012
3/16	0.0006	0.0006	0.0012	0.0026
5/16	0.0005	0.0005	0.0034	0.0063
7/16	0.0003	0.0003	0.008	0.012
9/16	0.0003	0.0003	0.0139	0.018
11/16	0.0001	0.0001	0.0187	0.0212
13/16	0.0001	0.0001	0.0193	0.0186
15/16	0.0001	0.0001	0.0123	0.0068

4. CONCLUSION

The problem of approximation of a transfer function of S.S. analysis has been treated to determine the solution via H.T. using a generalized algorithm. The derived algorithm is employed to solve for the state equation of the transfer function of a system via four different types of H.T.: $H^{\mathbb{I}}$,H^{II}, H^{III} and $H^{\mathbb{IV}}$ domains. Many tables are presented to compare the accuracies of different methods. From Table 1. Table 2. Table 3. Table 4. Table 5. Table 6. Table 7. Table 8. we conclude that the approximation of the system state is closer to the actual value in H.T. domain. It is noted that none of the presented methods proves itself absolutely superior to others, but from Table 3. Table 4. Table7. and Table 8. it is observed that the minimum percentage error is obtained for System state $\mathcal{X}_1(t)$ and $\mathcal{X}_2(t)$ for H.T of type $H^{\mathbb{I}}$ and $H^{\mathbb{III}}$domains-based computation. Finally, an advantage of H.T. based analysis is, H.T of type $H^{\mathbb{I}}$ and $H^{\mathbb{III}}$ based result may easily be obtained by simply dropping the System state $\mathcal{X}_1(t)$ and $\mathcal{X}_2(t)$solution of the H.T. based result. This advantage may prove much significant for function approximation as well as for control system analysis.

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