A Spectral-Collocation and Spline-Based Numerical Framework for Solving Nonlinear Boundary Layer Flow Problems

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ABSTRACT

In this work we present a multi-step algorithm with high accuracy to solve nonlinear second- and third-order ODE problems in fluid dynamics. In this method we use the polynomial spectral collocation method to acquire accurate numerical values at the collocation points, followed by spline interpolation to improve the approximated solution between the collocation points. Also, we derived quintic and septic polynomial spline models and computed the associated error bounds with a convergence analysis. To demonstrate the application of the present method, we considered two cases of fluid dynamics problems. We also looked at the numerical solution's validity and convergence in each example, which are assessed by calculating the infinity norm of the absolute and residual errors at every iteration level. Lastly, we illustrate the effect and influence of the physical parameters over velocity and temperature profiles through graphical results.

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1. INTRODUCTION

Numerous manufacturing processes, including the aerodynamic extrusion of paper production, plastic sheets, the drawing of plastic films, glass fiber, the cooling of metallic sheets in a cooling bath, and polymer sheets extruded continuously from a die, are illustrations of real-world uses for moving surfaces. Therefore, investigations of fluid dynamics problems are crucial.Nonlinear differential equations are a well-established mathematical topic, and their methodical development dates back to the early years of calculus's development. In heat transfer, boundary layer theory, and chemical reaction modeling, numerous scientific and technological issues are expressed as linear or nonlinear boundary value problems of second- and third-order ODEs with different kinds of boundary conditions. An incompressible electrically conducting fluid's steady hydromagnetic flow over an inclined stretching sheet [2] and the problem of a boundary layer flow over an unsteady stretching sheet in the presence of heat transfer and a Hall effect over the stretching surface [3] are two problems that are considered in this work. The extending surface and surrounding fluid move synchronously in various real-world situations. This includes applications such as cooling polymer sheets or films, cylinders, metallic sheets, etc. An exact solution is essential because the underlying differential equations controlling fluid motion in hydrodynamics contain a nonlinear component. It becomes challenging, if not impossible, to find the closed-form solutions. Consequently, most studies aim to approximate the solution. These kinds of issues have been researched and solved by numerous researchers, including [6], [8], [11], and [13]. Spectral approaches use global approximation functions (like high-order polynomials or the Fourier series) to describe the variable fields [7]. The spectral methods are thought to be the most precise approximation for smooth solutions due to their exponential convergence rate. Because computer memory was costly in the early days of CFD, spectral approaches were therefore frequently used in simulations.

The spline interpolation method for numerical analysis has been the subject of extensive research over the last few decades. To solve differential equations, several authors use spline techniques like [8], [17], and [18]. The quasi-linearization technique has gained popularity recently as a means of studying nonlinear problems [4]. This study solves heat transfer, hydromagneto dynamics, and boundary layer fluid flow problems, which are expressed as a system of nonlinear ordinary differential equations, by using a multi-step spline-quasi polynomial collocation and provides convergence analysis and numerical and graphical results.

1. MATERIAL AND METHODS:

In this section an ordinary equation of order n is considered that is given in the form Eq. (1) where $n = 2,3$. the procedure of constructing the solution algorithm for a single equation is shown in the upcoming subsections.

$$\tau\left[v, v', \dots, v^{(n)}\right] = g, \qquad x \in (a, b) \tag{1}$$

Here τ acting on v and its first n derivatives as a nonlinear operator, $v = v(x)$ is the unknown function, and $g(x)$ is any given function of x. The ODE Eq. (1) is solved subject to the given boundary conditions at $x = a$ and $x = b$, that are represented as.

$$\sum_{z=0}^{n-1} \alpha_s v^{(z)}(a) = w_{a,s,} \qquad s = 1,2, \dots, m_a \tag{2}$$

$$\sum_{z=0}^{n-1} \beta_r v^{(z)}(b) = w_{b,r,} \qquad r = 1,2, \dots, m_b \tag{3}$$

Where α_s and β_r are the constant coefficients of z-th ordinary derivative and $w_{a,s,}, w_{b,s,}$ are constants $v^{(z)}(x)$, Also m_a and m_b indicate the number of boundary conditions prescribed at $x = a$ and $x = b$ respectively. The method incorporates the steps described below.

2.1 The quasi-linearization method

Firstly, the nonlinear ODE is linearized using the quasi-linearization method (QLM) of bellman and Kalaba [4], using (QLM) for Eq. (1), we get.

$$\sum_{k=0}^{n} \pi_{k,l}(x) v_{l+1}^{(k)} = R_l(x) \tag{4}$$

Where $$\pi_{k,l}(x) = \frac{\partial \tau}{\partial v_l^{(k)}}\left[v_l, v_l', v_l'', \dots, v_l^{(n)}\right], \quad k = 0,1,2, \dots n$$

$$R_l(x) = g(x) + \sum_{k=0}^{n} \frac{\partial \tau}{\partial v_l^{(k)}}\left[v_l, v_l', v_l'', \dots, v_l^{(n)}\right] v_l^{(k)} - \tau\left[v_l, v_l', v_l'', \dots, v_l^{(n)}\right]$$

Here $l = 1,2,3 \dots$, shows the iteration stage, beginning with an initial approximate solution v_0, that is found from the boundary conditions Eq. (2,3), The QLM scheme is used to solve Eq. (4) iteratively until the required accuracy tolerance is obtained.

2.2 Spectral collocation

The spectral quasi-linearization method is implemented on the linearized QLM scheme Eq. (4) as shown above, the approximate solution is assumed to be a polynomial of degree $N + n$.

$$v(x) \approx V(x) = \sum_{k=0}^{n+N} c_k x^k \tag{5}$$

where n is order of the ODE and the domain $[a, b]$ is partitioned to N subintervals so that we have $N + 1$ collocation points and c_ks are constants to be determined.

2.3 Spline Interpolation

The linearized ODE in Eq. (4) is solved using the proposed model in Eq. (5), the values of $v_i, v_i', \dots, v_i^{(n)}$ are obtained at each x_i, these accurate approximated values are used to refine the solution using the spline model illustrated below.

$$S_{i,2n+1}(x) = \sum_{r=0}^{n} \frac{b_r(x-x_i)^r}{r!} + \sum_{s=n+1}^{2n+1} \frac{\alpha_s(x-x_i)^{(s)}}{s!} \tag{6}$$

Where $x \in [x_i, x_{i+1}]$ and α_s are unknowns to be determined. To find the values of α_s we substitute x with x_i so that.

$$S_{i,2n+1}^{(r)}(x_i) = v_i^{(r)} \text{ where } r = 0,1,\dots,n.$$

We get $b_r = \frac{v_i^{(r)}}{r!}$ then we use Taylor series expansion around x_i for $v, v', \dots, v^{(n)}$ and substitute x with x_{i+1} so that.

$$S_{i,2n+1}^{(r)}(x_{i+1}) = v_{i+1}^{(r)} \tag{7}$$

Different polynomial spline models of various degrees are used depending on the order of the ODE in Eq. (4) as shown in the next subsection.

2.3.2 Construction of Quintic Spline.

Here we consider an ODE in Eq.(1) where $n = 2$, we construct a polynomial spline of degree five where y_i, y_i' and y_i'' are accurate approximations for v_i, v_i' and v_i'', using continuity conditions for splines the following model can be derived.

$$S_{i,5}(x) = c_{i,2}(x-x_i)^5 + c_{i,1}(x-x_i)^4 + c_{i,0}(x-x_i)^3 + \frac{y_i''}{2}(x-x_i)^2 + y_i'(x-x_i) + y_i \tag{8}$$

Using continuity conditions in Eq.(7), the unknowns $c_{i,0}, c_{i,1}$ and $c_{i,2}$ are found by solving the system of equations below.

$$h^5 c_{i,2} + h^4 c_{i,1} + h^3 c_{i,0} = y_{i+1} - y_i - h y_i' - \frac{h^2}{2} y_i'' \tag{9}$$

$$5h^4 c_{i,2} + 4h^3 c_{i,1} + 3h^2 c_{i,0} = y_{i+1}' - y_i' - h y_i'' \tag{10}$$

$$20h^3 c_{i,2} + 12h^2 c_{i,1} + 6h c_{i,0} = y_{i+1}'' - y_i'' \tag{11}$$

As a result, we obtain.

$c_{i,2} = \frac{1}{2h^5}[-12 \quad 12 \quad -6 \quad -6 \quad -1 \quad 1] * Y_2$, $c_{i,1} = \frac{1}{2h^4}[30 \quad -30 \quad 16 \quad 14 \quad 3 \quad -2] * Y_2$

and $c_{i,0} = \frac{1}{2h^3}[-20 \quad 20 \quad -12 \quad -8 \quad -3 \quad 1] * Y_2$, where

$$Y_2 = [y_i \quad y_{i+1} \quad h y_i' \quad h y_{i+1}' \quad h^2 y_i'' \quad h^2 y_{i+1}'']^T \tag{12}$$

1.3.3 Construction of Septic Spline.

Here we consider an ODE in Eq.(1) where $n = 3$. We aim to construct a polynomial spline of degree seven where y_i, y_i', y_i'' and $y_i^{(3)}$ are accurate approximations for v_i, v_i', v_i'' and $v_i^{(3)}$. Using continuity conditions for spline functions, the following model can be derived.

$$S_{i,7}(x) = b_{i,3}(x-x_i)^7 + b_{i,2}(x-x_i)^6 + b_{i,1}(x-x_i)^5 + b_{i,0}(x-x_i)^4 + \frac{y_i^{(3)}}{6}(x-x_i)^3 + \frac{y_i''}{2}(x-x_i)^2 + y_i'(x-x_i) + y_i \tag{13}$$

Using continuity conditions for Eq. (13), the unknowns $b_{i,0}, b_{i,1}, b_{i,2}$ and $b_{i,3}$ are found by solving the system of equations below.

$$h^7 b_{i,3} + h^6 b_{i,2} + h^5 b_{i,1} + h^4 b_{i,0} = y_{i+1} - y_i - h y_i' - \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i^{(3)} \tag{14}$$

$$7h^6 b_{i,3} + 6h^5 b_{i,2} + 5h^4 b_{i,1} + 4h^3 b_{i,0} = y_{i+1}' - y_i' - h y_i'' - \frac{h^2}{2} y_i^{(3)} \tag{15}$$

$$42h^5 b_{i,3} + 30h^4 b_{i,2} + 20h^3 b_{i,1} + 12h^2 b_{i,0} = y''_{i+1} - y''_i - hy_i^{(3)} \quad (16)$$

$$210h^4 b_{i,3} + 120h^3 b_{i,2} + 60h^2 b_{i,1} + 24hb_{i,0} = y_{i+1}^{(3)} - y_i^{(3)} \quad (17)$$

We get $b_{i,3} = \frac{1}{6h^7}[120 \quad -120 \quad 60 \quad 60 \quad 12 \quad -12 \quad 1 \quad 1] * Y_3,$

$b_{i,2} = \frac{1}{6h^6}[-420 \quad 420 \quad -216 \quad -204 \quad -45 \quad 39 \quad -4 \quad -3] * Y_3,$

$b_{i,1} = \frac{1}{2h^5}[168 \quad -168 \quad 90 \quad 78 \quad 20 \quad -14 \quad 2 \quad 1] * Y_3$, and

$$b_{i,0} = \frac{1}{6h^4}[-210 \quad 210 \quad -120 \quad -90 \quad -30 \quad 15 \quad 4 \quad 1] * Y_3$$

Where $Y_3 = [y_i \quad y_{i+1} \quad hy'_i \quad hy'_{i+1} \quad h^2y''_i \quad h^2y''_{i+1} \quad h^3y_i^{(3)} \quad h^3y_{i+1}^{(3)}]^T$ (18)

2. CONVERGENCE ANALYSIS

In this part the validity and accuracy of the proposed spline models are assessed. Firstly, we prove some lemmas by finding the error bounds of the coefficients in Eq. (12,18) and check the convergence of the proposed models in equations (8,13) by proving error bound theorems, Similar to some previous efforts that examined the convergence of some types of spline functions [10], and [17].

Lemma 3.1:Let $y_i(x)$ be spline polynomial interpolation of a function $y(x)$ on the interval $[x_i, x_{i+1}]$ using $y(x) \in C^6[0,1]$, then the following inequalities hold.

$$\left|(3+q)!\, c_{i,q} - y_i^{(3+q)}\right| \le h^{3-q}\left(\alpha_q y^{(4)}(\xi_1) + \beta_q y^{(4)}(\xi_2) + \varphi_q y^{(6)}(\xi_3)\right),$$

where $\xi_1, \xi_2, \xi_3 \in (x_i, x_{i+1})$,

$\alpha_2 = 1, \alpha_1 = -\frac{1}{2}, \alpha_0 = \frac{1}{12}, \beta_2 = -3, \beta_1 = \frac{14}{10}, \beta_0 = -\frac{1}{5}, \varphi_2 = \frac{5}{2}, \varphi_1 = -1, \varphi_0 = \frac{1}{8}$ for $q = 0,1,2$.

Proof. Using Taylor series expansion for $y \in C^6[0,1]$, about x_i and the results from Eq. (12)

We get $\left|5!\, c_{i,2} - y_i^3\right| \le \frac{h}{2}\left(2y^{(6)}(\xi_1) - 6y^{(6)}(\xi_2) + 5y^{(6)}(\xi_3)\right),$

$|4!\, c_{i,1} - y''_i| \le \frac{h^2}{10}\left(-5y^{(6)}(\xi_1) + 14y^{(6)}(\xi_2) - 10y^{(6)}(\xi_3)\right),$

And $|3!\, c_{i,0} - y''_i| \le \frac{h^3}{120}\left(10y^{(6)}(\xi_1) - 24y^{(6)}(\xi_2) + 15y^{(6)}(\xi_3)\right)$

Theorem 3.2: Let $S_i(x)$ be spline polynomial interpolation which satisfies Eq. (7), where $r = 0,1,2$ and $y \in C^6[0,1]$, such that $x \in (x_i, x_{i+1})$ and $h = (x_{i+1} - x_i)$, then

$$\left|y^q(x) - S_{i,5}^{(q)}(x)\right| \le C_q h^{6-q} \omega\left(y^{(6)}, h\right)$$

where $C_0 = \frac{1}{7!}, C_1 = \frac{1}{6!}, C_2 = \frac{1}{5!}, C_3 = \frac{23}{60}, C_4 = \frac{21}{10}, C_5 = 4$,

and $C_6 = 1$, where $q = 0, 1, 2, 3, 4, 5, 6$.

Proof. Using Taylor series expansion for $y \in C^6[0,1]$, about x_i and Lemma 3.1, then we get

$$\left|y^{\square}(x) - S_{i,5}^{\square}\right| = \frac{h^6}{6!}\left|y^{(6)}(\xi_1) - y^{(6)}(\xi_4)\right| \le \frac{h^6}{6!}\omega\left(y^{(6)}; h\right)$$

$$\left|y'(x)-S'_{i,5}(x)\right|=\frac{h^5}{5!}\left|y^{(6)}(\xi_2)-y^{(6)}(\xi_4)\right|\le \frac{h^5}{5!}\omega\left(y^{(6)};h\right)$$

$$\left|y''(x)-S''_{i,5}(x)\right|=\frac{h^5}{4!}\left|y^{(6)}(\xi_3)-y^{(6)}(\xi_4)\right|\le \frac{h^4}{4!}\omega\left(y^{(6)};h\right)$$

$$\left|y^{(3)}(x)-S^{(3)}_{i,5}(x)\right|=\frac{h^3}{120}\left|10y^{(4)}(\xi_1)-36y^{(4)}(\xi_2)+45y^{(4)}(\xi_3)-20y^{(6)}(\xi_4)\right|\le \frac{23h^3}{60}\omega\left(y^{(6)};h\right)$$

$$\left|y^{(4)}(x)-S^{(4)}_{i,5}(x)\right|=\frac{h^2}{10}\left|5y^{(4)}(\xi_1)-16y^{(4)}(\xi_2)+15y^{(4)}(\xi_3)-5y^{(6)}(\xi_4)\right|\le \frac{21h^2}{10}\omega\left(y^{(6)};h\right)$$

$$\left|y^{(5)}(x)-S^{(5)}_{i,5}(x)\right|=\frac{h^2}{2}\left|2y^{(4)}(\xi_1)-6y^{(4)}(\xi_2)+5y^{(4)}(\xi_3)-2y^{(6)}(\xi_4)\right|\le 4h\omega\left(y^{(6)};h\right)$$

$$\left|y^{(6)}(x)-S^{(6)}_{i,5}(x)\right|\le \omega\left(y^{(6)};h\right)$$

Lemma3.3: Let $y_i(x)$, be spline polynomial interpolation of a function $y(x)$ on the interval $[x_i, x_{i+1}]$ then

$$\left|(4+q)!\,c_{i,q}-y_i^{(4+q)}\right|\le h^{4-q}\left(\alpha_q y^{(4)}(\xi_1)+\beta_q y^{(4)}(\xi_2)+\varphi_q y^{(6)}(\xi_3)+\emptyset_q y^{(6)}(\xi_4)\right),$$

where $\xi_1, \xi_2, \xi_3, \xi_3 \in (x_i, x_{i+1})$,

$\alpha_3=-\frac{5}{2}, \alpha_2=\frac{35}{28}, \alpha_1=-\frac{21}{84}, \alpha_0=\frac{35}{1680}, \beta_3=10, \beta_2=-\frac{34}{7}, \beta_1=\frac{39}{41}, \beta_0=-\frac{3}{41}, \varphi_3=-14, \varphi_2=\frac{91}{14}, \varphi_1=-\frac{49}{42}, \varphi_0=\frac{1}{12}$ $\emptyset_3=7, \emptyset_2=-3, \emptyset_1=\frac{1}{2}, \emptyset_0=\frac{23}{840}$, for $q=0,1,2,3$.

Proof. Using Taylor series expansion for $y(x)\in C^8[0,1]$, about x_i and the results from Eq. (18), we have

$$\left|7!\,b_{i,3}-y_i^{(7)}\right|\le \frac{h}{2}\left(-5y^{(8)}(\xi_1)+20y^{(8)}(\xi_2)-28y^{(8)}(\xi_3)+14y^{(8)}(\xi_4)\right)$$

$$\left|6!\,b_{i,2}-y_i^{(6)}\right|\le \frac{h^2}{28}\left(35y^{(8)}(\xi_1)-136y^{(8)}(\xi_2)+182(\xi_3)-84y^{(8)}(\xi_4)\right)$$

$$\left|5!\,b_{i,3}-y_i^{(5)}\right|\le \frac{h^3}{84}\left(-21y^{(8)}(\xi_1)+78y^{(8)}(\xi_2)-98y^{(8)}(\xi_3)+42y^{(8)}(\xi_4)\right)$$

$$\left|4!\,b_{i,3}-y_i^{(4)}\right|\le \frac{h^4}{1680}\left(35y^{(8)}(\xi_1)-120y^{(8)}(\xi_2)+140y^{(8)}(\xi_3)-56y^{(8)}(\xi_4)\right)$$

Theorem 3.4: Let $S_i(x)$ be spline interpolation which satisfies Eq.(7), where $r=0,1,2,3$ and $y\in C^8[0,1]$, such that $x_i<x<x_{i+1}$ and $h=(x_{i+1}-x_i)$, then

$\left|y^q(x)-S^{(q)}_{i,7}\right|\le B_q h^{8-q}\omega\left(y^{(8)},h\right)$, where $B_0=\frac{1}{40320}, B_1=\frac{1}{5040}, B_2=\frac{1}{720}, B_3=\frac{1}{120}, B_4=\frac{385}{1680}, B_5=\frac{175}{84}$, $B_6=\frac{259}{28}$ and $B_7=\frac{35}{2}, B_8=1$, and q=0, 1, 2, 3, 4, 5, 6, 7, 8.

Proof. Using Taylor series expansion for $y\in C^8[0,1]$, about x_i and Lemma 3.3, then we obtain.

$$\left|y(x)-S_{i,7}\right|=\frac{h^8}{8!}\left|y^{(8)}(\xi_1)-y^{(8)}(\xi_5)\right|\le \frac{h^8}{8!}\omega\left(y^{(8)};h\right)$$

$$\left|y'(x)-S'_{i,7}(x)\right|=\frac{h^7}{7!}\left|y^{(8)}(\xi_2)-y^{(8)}(\xi_5)\right|\le \frac{h^7}{7!}\omega\left(y^{(8)};h\right)$$

$$\left|y''(x)-S''_{i,7}(x)\right|=\frac{h^6}{6!}\left|y^{(8)}(\xi_3)-y^{(8)}(\xi_5)\right|\le \frac{h^6}{6!}\omega\left(y^{(8)};h\right)$$

$$\left|y^{(3)}(x)-S^{(3)}_{i,7}(x)\right|=\frac{h^5}{5!}\left|y^{(8)}(\xi_4)-y^{(8)}(\xi_5)\right|\le \frac{h^5}{5!}\omega\left(y^{(8)};h\right)$$

$$\left|y^{(4)}(x)-S_{i,7}^{(4)}(x)\right|=\frac{h^4}{1680}\left|-35y^{(8)}(\xi_1)+160y^{(8)}(\xi_2)-280y^{(8)}(\xi_3)+224y^{(8)}(\xi_4)-70y^{(8)}(\xi_5)\right|\leq \frac{385h^4}{1680}\omega\left(y^{(6)};h\right)$$

$$\left|y^{(5)}(x)-S_{i,7}^{(5)}(x)\right|=\frac{h^3}{84}\left|-21y^{(8)}(\xi_1)+90y^{(8)}(\xi_2)-140y^{(8)}(\xi_3)+84y^{(8)}(\xi_4)-14y^{(8)}(\xi_5)\right|\leq \frac{175h^4}{84}\omega\left(y^{(6)};h\right)$$

$$\left|y^{(6)}(x)-S_{i,7}^{(6)}(x)\right|=\frac{h^2}{28}\left|-35y^{(8)}(\xi_1)+144y^{(8)}(\xi_2)-210y^{(8)}(\xi_3)+112y^{(8)}(\xi_4)-14y^{(8)}(\xi_5)\right|\leq \frac{259h^2}{28}\omega\left(y^{(8)};h\right)$$

$$\left|y^{(7)}(x)-S_{i,7}^{(7)}(x)\right|=\frac{h}{2}\left|-5y^{(8)}(\xi_1)+20y^{(8)}(\xi_2)-28y^{(8)}(\xi_3)+14y^{(8)}(\xi_4)-2y^{(8)}(\xi_5)\right|\leq \frac{35h}{2}\omega\left(y^{(8)};h\right)$$

$$\left|y^{(8)}(x)-S_{i,7}^{(8)}(x)\right|=\omega\left(y^{(8)};h\right)$$

3. NUMERICAL EXPRIMENTATION

In this part, we consider some boundary layer problems that can be defined on truncated domain, the aim of this section is to show the accuracy and validity of our developed method.

Example 1. A problem of a laminar two-dimensional steady boundary layer flow of an incompressible viscous dusty fluid over a vertical stretching sheet is considered. The governing equations are written in similarity form in [2] as.

$$f^3+ff''-f'^2+G_r\cos\gamma\theta-Qf'=0, \tag{19}$$

$$\theta''+P_rf\theta'-P_rmf'\theta+P_rMf'^2+P_rE_cf''^2=0, \tag{20}$$

The boundary conditions are given by

$$f'(0)=1,\ f(0)=0,\theta(0)=1,\theta(\infty)=0,\ f'(\infty)=0 \tag{21}$$

Applying QLM we get.

$$f_{l+1}^{(3)}+f_lf_{l+1}''+[-2f_l'-Q]f_{l+1}'+f_l''f_{l+1}+G_r\cos\gamma\theta_{l+1}=f_l''f_l-f_l'^2 \tag{22}$$

$$\theta_{l+1}''+P_rf_l\theta_{l+1}'+[-P_rmf_l']\theta_{l+1}+[2P_rE_cf_l'']f_{l+1}''+[-P_rm\theta_l+2P_rMf_l]f_{l+1}'+P_r\theta_l'f_{l+1}= P_rMf_l'^2+P_rMf_l''^2+P_rm\theta_lf_l'+P_r\theta_l'f_l \tag{23}$$

Subject to the boundary conditions $f_{l+1}(0)=0, f_{l+1}'(0)=1, \theta_{l+1}(0)=1, f_{l+1}'(\infty)=0, \theta_{l+1}(\infty)=0$ (24)

In this system of ODE, the accuracy of the scheme is examined by evaluating the residual error at the present iteration $(l+1)$, which can be described as.

$\mathrm{Res}_F=|F_{l+1}|_\infty, \mathrm{Res}_\theta=|\Theta_{l+1}|_\infty$, where $F_{l+1}=f_{l+1}^3+f_{l+1}f_{l+1}''-f'^2_{l+1}+G_r\cos\gamma\theta_{l+1}-Qf_{l+1}'$, and

$$\Theta_{l+1}=\theta_{l+1}''+P_rf_{l+1}\theta_{l+1}'-P_rmf_{l+1}'\theta_{l+1}+P_rM(f_{l+1}')^2+P_rE_cf''^2_{l+1} \tag{25}$$

Additionally the scheme is said to be convergent if $|f_{l+1}-f_l|<\epsilon_f, |\theta_{l+1}-\theta_l|\theta<\epsilon_\theta, \forall l>L$, for some chosen tolerances ϵ_f and ϵ_θ.

Example 2. Lastly, we consider the boundary layer flow over an unsteady stretching sheet in the presence of the Hall effect and heat transfer over a stretching surface; the existing systems are expressed in their non-dimensional form by EL-Aziz [3] as

$$f^{(3)} + ff'' - (f)'^2 - A\left(f' + \frac{xf''}{2}\right) - \frac{M}{1+m^2}(mh + f') = 0 \quad (26)$$

$$h'' + fh' - f'h - A\left(h + \frac{x}{2}h'\right) + \frac{M}{1+m^2}(mf' + h) = 0 \quad (27)$$

$$\frac{1}{Pr}\theta'' + f\theta' - 2f'\theta - \frac{A}{2}(3\theta + x\theta') = 0 \quad (28)$$

Dependent on the boundary conditions

$$f(0) = 0,\ f'(0) = 1,\ h(0) = 0,\ \theta(0) = 1,\ f'(\infty) = 0,\ h(\infty) = 0,\ \theta(\infty) = 0 \quad (29)$$

In this problem, $f'(x), h(x)$ and $\theta(x)$ are unknown functions of that are representing the transverse velocity, the dimensionless temperature, and the axial velocity, respectively; m is the Hall Effect parameter, the unsteadiness parameter is A, the Prandtl number is Pr, and lastly, M is the magnetic parameter.

The QLM scheme for the governing system of the ODEs Eq. (26-29), is given by

$$f_{l+1}^{(3)} + \left(f_l - \frac{Ax}{2}\right)f''_{l+1} + \left(-A - 2f'_l - \frac{M}{1+m^2}\right)f'_{l+1} + f_l''f_{l+1} - \frac{mM}{1+m^2}h_{l+1} = f'_l f''_l - f_l'^2 \quad (30)$$

$$\left(-h_l + \frac{mM}{1+m^2}\right)f'_{l+1} + h'_l f_{l+1} + h''_{l+1} + \left(f_l - \frac{Ax}{2}\right)h'_{l+1} + \left(-2f'_l - A - \frac{M}{1+m^2}\right)h_{l+1} = f_l h'_l - f'_l h_l \quad (31)$$

$$-2\theta_l f'_{l+1} + \theta'_l f_{l+1} + \frac{1}{Pr}\theta''_{l+1} + \left(f_l - \frac{Ax}{2}\right)\theta'_{l+1} + \left(-2f'_l - \frac{3A}{2}\right)\theta_{l+1} = f_l\theta'_l - 2\theta_l f'_l \quad (32)$$

The boundary conditions are given below as

$$f_{l+1}(0) = 0,\ f'_{l+1}(0) = 1, h_{l+1}(0) = 0,\ \theta_{l+1}(0) = 1,\ f'_{l+1}(\infty) = 0, h_{l+1}(\infty) = 0, \theta_{l+1}(\infty) = 0 \quad (33)$$

Similarly to the previous examples, the validity and precision of the iterative system in Eq. (30-33) are tested through residual error analysis at the present iteration $l + 1$ and are described as follows.

$Res_F = |F_{l+1}|_\infty, Res_H = |H_{l+1}|_\infty$ and $Res_\theta = |\Theta_{l+1}|_\infty$ where

$$F_{l+1} = f_{l+1}^{(3)} + f_{l+1}f''_{l+1} - (f_{l+1})'^2 - A\left(f'_{l+1} + \frac{x}{2}f''_{l+1}\right) - \frac{M}{1+m^2}(f'_{l+1} + mh_{l+1}), \quad (34)$$

$$H_{l+1} = h''_{l+1} + f_{l+1}h'_{l+1} - f'_{l+1}h_{l+1} - A\left(h_{l+1} + \frac{x}{2}h'_{l+1}\right) + \frac{M}{m^2+1}(h_{l+1} + mf'_{l+1}) \quad (35)$$

$$\Theta_{l+1} = \frac{1}{Pr}\theta''_{l+1} + f_{l+1}\theta'_{l+1} - 2f'_{l+1}\theta_{l+1} - \frac{A}{2}(3\theta_{l+1} + x\theta'_{l+1}) \quad (36)$$

Additionally the scheme is said to be convergent if $|f_{l+1} - f_l| < \epsilon_f, |\theta_{l+1} - \theta_l| < \epsilon_\theta$ and $|h_{l+1} - h_l| < \epsilon_h\ \forall l > L$, for some chosen tolerances ϵ_f, ϵ_h and ϵ_θ .

4. Numerical results and discussions

In this part, to solve the linear iterative equations Eq. (22-24) and Eq. (30-33), the solutions are approximated using smooth polynomial spline segments. The residual error is computed for each nonlinear operator at each iteration to demonstrate the convergence and efficacy of our method. In Table (1), the residual error for Eq. (22-24) is evaluated for three iterations. Also in Table (2), the residual errors of Eq. (30-33) are evaluated for 5 iterations given in Eq. (34-36); hereafter, comparing the results with [9]. Additionally, to examine the convergence and the stability of our proposed method numerically, the infinity norm is computed. Table (3) and Table (4) show that the iterative method in Eq. (22-24) and Eq. (30-33) converges after 6 or 7 iterations. Figure (1-4) discusses the changes in velocity profile f' and non-dimensional temperature θ in Eq. (19-21) under the influence of the magnetic parameter M, the Prandtl number Pr, the angle of inclination γ, the Chandrasekhar number Q, and the Eckert number Ec. In Figure (1), a decrease in values of θ is observed when the values of M and Pr increase. Figure (2) shows that increasing the values of γ and Q leads to an increment in f'.

However, Figure (3) suggests that raising the values of Q and γ results in an increment in the first section of θ domain and a decrease in the values of θ in its second section. It is also noticeable from Figure (4) that the values of $f'$and θ vary directly with Ec. Figure (5-8) shows the variation of axial velocity f', transverse velocity h, and non-dimensional temperature θ in Eq. (26-29) under the effect of the magnetic parameter M, the Prandtl number Pr, the Hall effect parameter A, and the unsteadiness parameter m. From Figure (5), it is observable that increasing the value of Pr and the Hall effect parameter A results in a decrease in the dimensionless temperature θ. Also, raising the values of the Hall effect parameter A leads to a significant decline in the values of $f'$and h, which can be seen in Figure (6). In Figure (7) it can be noticed that by increasing the values of the unsteadiness parameter m, the values of f' also increase, but raising the values of the magnetic parameter M results in an increment in the values of f' Lastly, it can be seen from Figure (8) that increasing the values of the unsteadiness parameter m decreases the values of h, and raising the values of the magnetic parameter M causes a significant increment in h values.

Table 1. Residual error analysis of Spline approximations in Example 1 on the finite domain [0,1] using $Q = 0.8, G_r = 5, \gamma = 60°, m = 1, M = 1, P_r = 1$ and $E_c = 1$

Iteration	Residual error	
L	$\|F_{l+1}\|_\infty$,N = 8	$\|\Theta_{l+1}\|_\infty$,N = 8
1	2.2932815e-002	5.0869839e-001
2	2.5396168e-007	2.1834309e-006
3	2.6413052e-017	4.1447793e-016

Table 2. Comparison of the residual error for Eq. (30-33) in Example 2 between the present work and [9].

Iteration	$\|F_{l+1}\|_\infty$	
L	[9]	Present work
1	3.75070e-001	1.00974e+00
2	3.58227e-002	2.45008e-002
3	2.58387e-004	3.61489e-006
4	1.01330e-008	3.48240e-014
5	3.09963e-012	1.99649e-030
Iteration	$\|H_{l+1}\|_\infty$	
L	[9]	Present work
1	5.67287e-002	2.19961e-002
2	9.17907e-004	9.09012e-004
3	2.26789e-005	2.48135e-007
4	1.70642e-009	3.56051e-015
5	1.16504e-013	2.82517e-031
Iteration	$\|\Theta_{l+1}\|_\infty$	
L	[9]	Present work
1	5.11974e-002	2.32423e-001
2	1.56959e-002	5.25913e-003
3	3.87065e-004	1.16030e-006
4	6.96569e-008	1.82589e-014
5	5.87419e-013	1.77721e-030

Table 3. Numerical Convergence analysis of Spline approximations using 11 points ($N = 10$) for Eq. (19-20) in Example 1.

Iteration	Absolute Error	
L	$\|f_{l+1} - f_l\|_\infty$	$\|\theta_{l+1} - \theta_l\|_\infty$
1	2.321519e-02	8.132948e-02
2	5.769752e-05	4.283432e-04
3	7.453655e-10	9.448256e-09
4	1.650832e-19	4.917336e-19
5	1.102026e-39	2.589761e-38
6	5.510130e-40	3.673420e-40
7	7.346840e-40	3.673420e-40

Table 4. Numerical Convergence analysis of Spline approximations using 11 points ($N = 10$) for Eq. (26-29) in Example 2.

Iteration	Absolute Error		
l	$\lvert f_{l+1} - f_l \rvert_\infty$	$\lvert h_{l+1} - h_l \rvert_\infty$	$\lvert \theta_{l+1} - \theta_l \rvert_\infty$
1	1.359112e-02	2.255401e-02	1.072987e-01
2	6.323724e-02	4.904691e-03	2.078343e-02
3	6.185082e-04	8.483846e-05	3.509774e-04
4	5.398827e-08	1.103315e-08	5.164501e-08
5	3.832820e-16	1.072942e-16	6.102415e-16
6	1.850823e-32	6.788643e-33	4.871858e-32
7	5.510130e-40	7.461634e-41	3.673420e-40
8	4.591775e-40	9.183550e-41	3.673420e-40

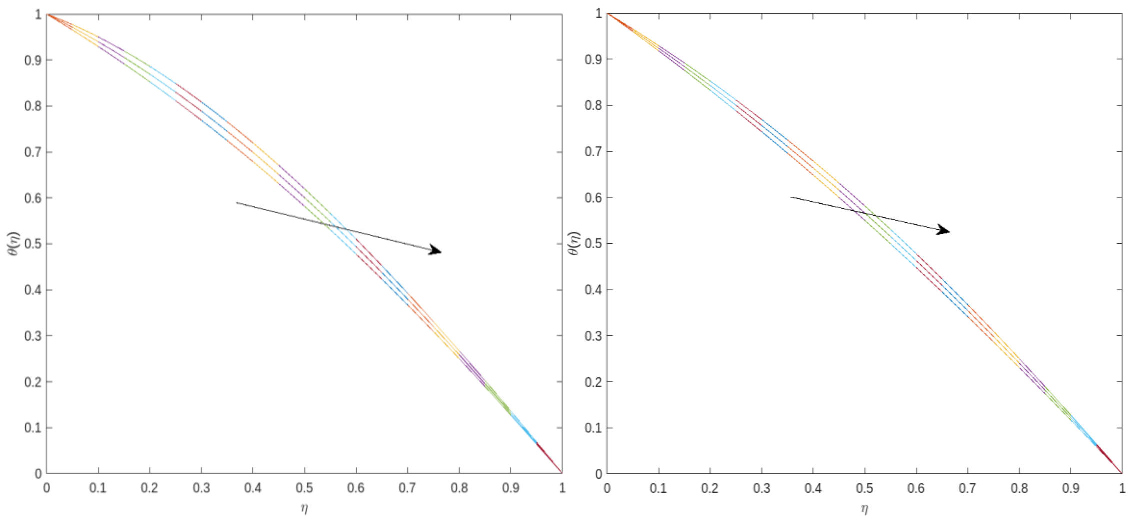

Figure 1. The change of temperature $\theta(\eta)$ profile under the effect of magnetic parameter M and Prandtl number Pr, when $M = 1, 1.5, 2$ and $Pr = 0.6, 0.8, 1$.

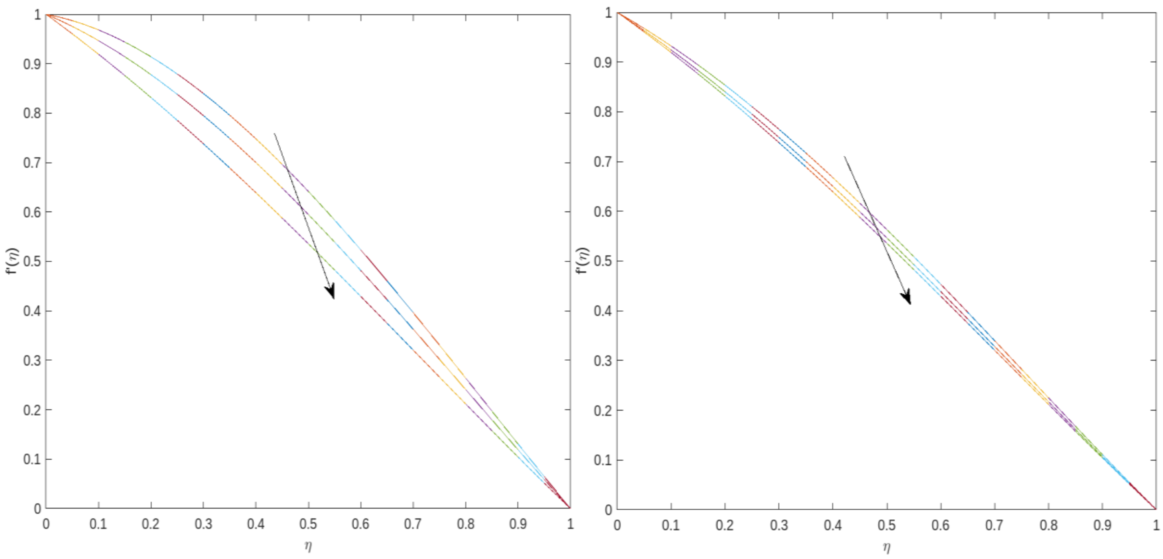

Figure 2. Impact of γ the angle of inclination and Chandrasekhar number Q on non-dimensional velocity $f'(\eta)$ for $\gamma = 30°, 45°, 60°$, and $Q = 0.3, 0.6, 0.8$.

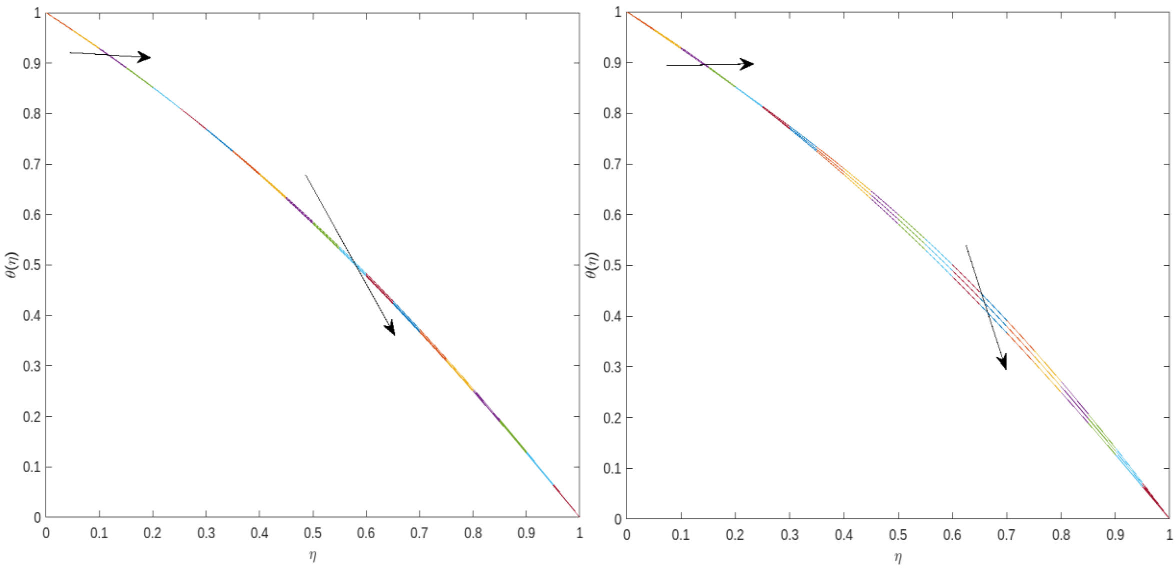

Figure 3. Effect of γ the angle of inclination and Chandrasekhar number Q on the temperature profile $\theta(\eta)$, when $\gamma = 30°, 45°, 60°$, and $Q = 0.3, 0.6, 0.8$.

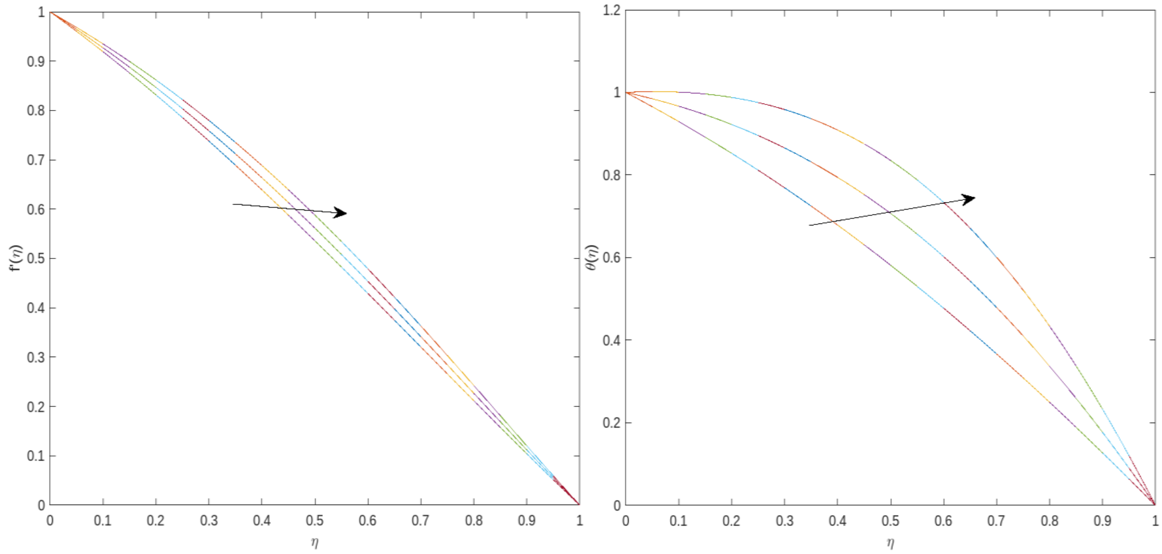

Figure 4. Influence of Eckert number Ec on non-dimensional $f'(\eta)$ velocity and temperature $\theta(\eta)$ profile for $Ec = 1, 2, 3$.

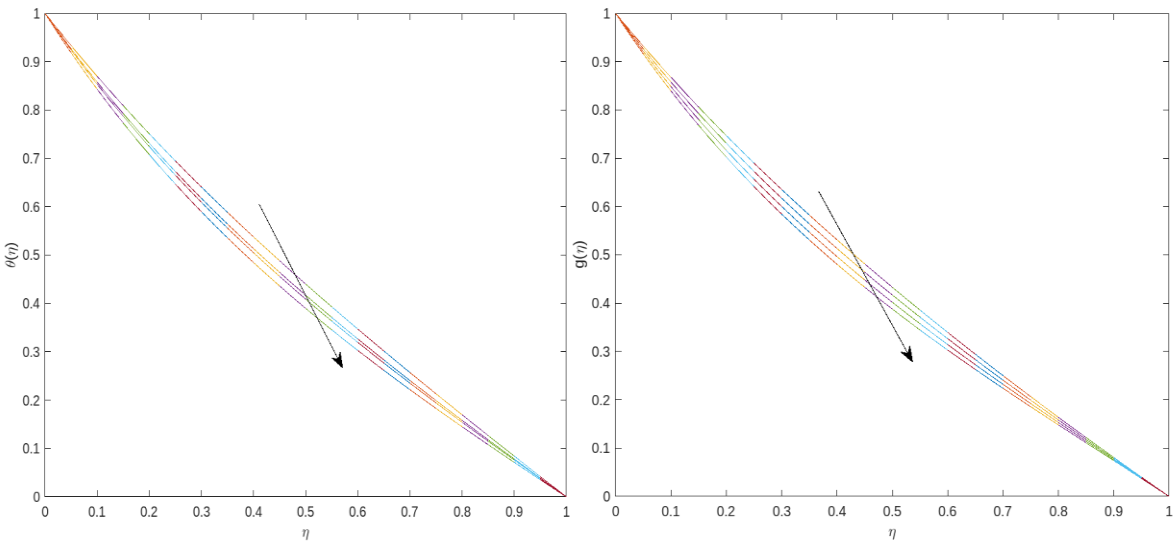

Figure 5. Variation of temperature profile $\theta(\eta)$ under the influence of Prandtl number Pr and the Hall effect parameter A for $A = 0, 0.5, 1, 1.5$ and $Pr = 0.5, 0.72, 0.8, 1$.

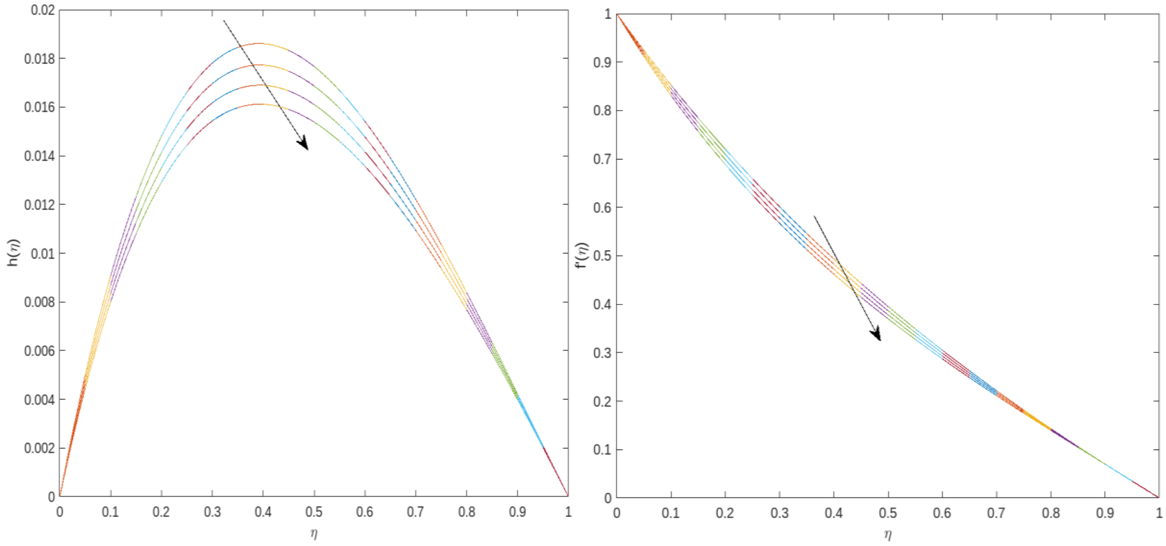

Figure 6. Graphical representation of the transverse velocity $h(\eta)$ and axial velocity $f'(\eta)$ profile under the influence of the Hall effect parameter A, when $A = 0, 0.5, 1, 1.5$.

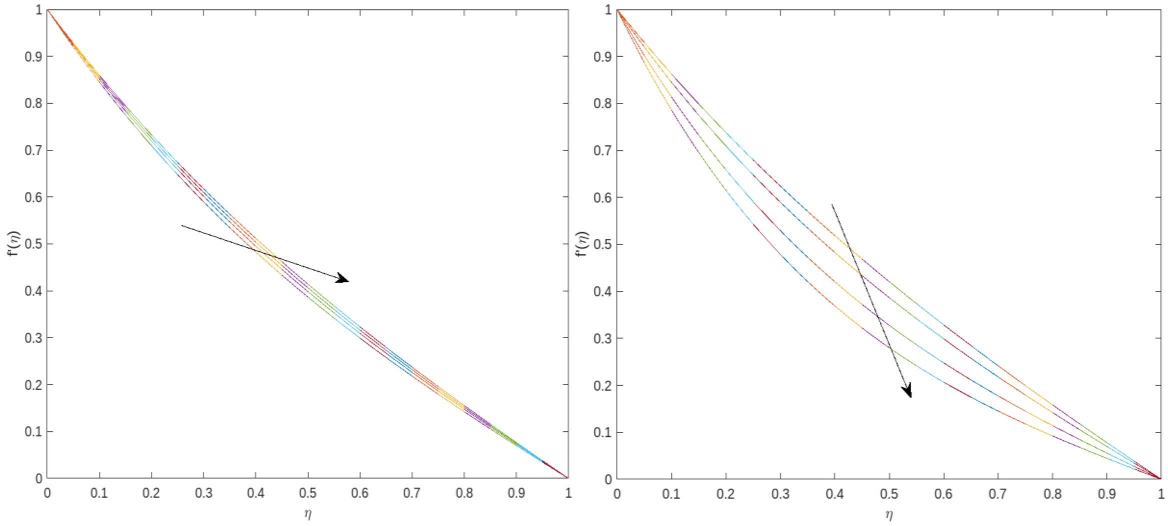

Figure 7. Effect of unsteadiness parameter m and the magnetic parameter M on the axial velocity $f'(\eta)$ for $m = 0.5, 1, 1.5, 2.5$, and $M = 0, 1, 3, 5$.

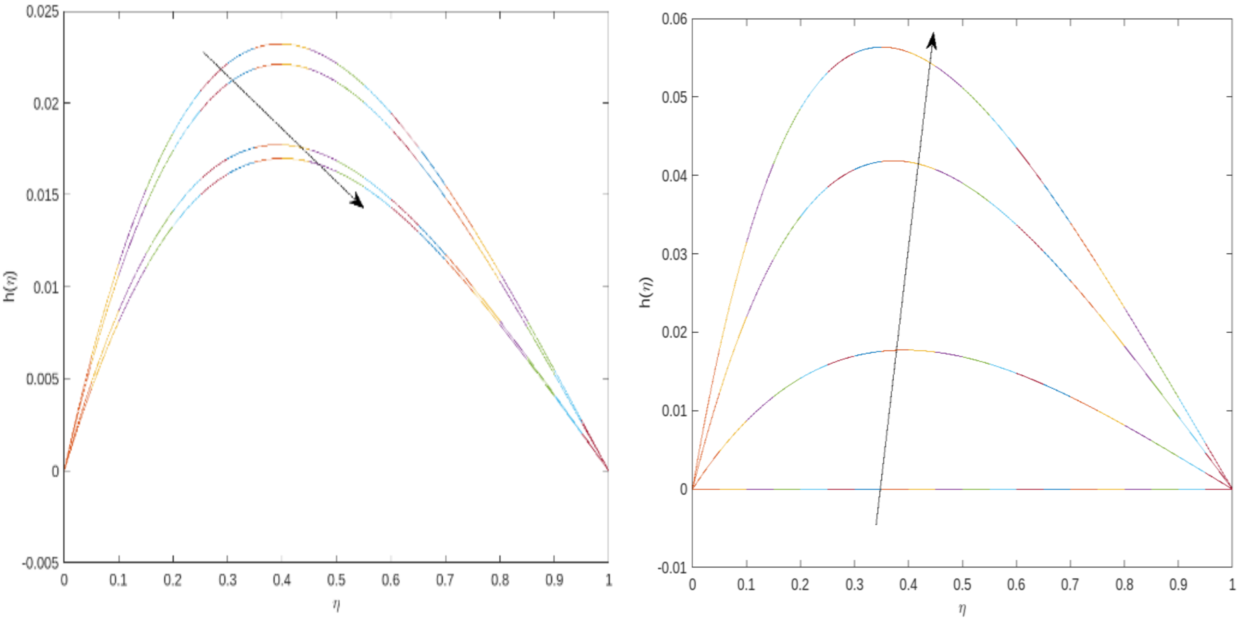

Figure 8. The graph of dimensionless transverse velocity $h(\eta)$ under the effect of the unsteadiness parameter m and the magnetic parameter M, when $m = 0.5, 1, 1.5, 2.5$ and $M = 0, 1, 3, 5$.

5. CONCLUSIONS

This paper presents an effective numerical technique for solving nonlinear boundary layer flow problems using spectral spline collocation schemes, which are introduced together with the use of the quasi-linearization method, and the solution curve is spatially interpolated using the functions in the Matlab program. We obtain the maximum errors between each pair of consecutive iterative solutions, and residual errors are computed. The numerical approach shown in the Table (1-4) at the uniform points with respect to the step size h, and comparing the scheme suggested the method to others already established in the literature [2] [3] [9]. It offers sufficient precision and is innovative. Additionally, two numerical cases have been examined utilizing the spectral spline approach, and graphs show the accuracy and practicality of the approach. Finally, we strongly advise applying comparable multi-step techniques to solve fluid dynamic issues by utilizing spectral methods and different spline functions.

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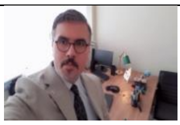	**Dr. Shko Ali Tahir**, Lecturer of Numerical Analysis. Department of Mathematics; College of Sciences; University of Sulaimani. Email : shko.tahir@univsul.edu.iq