Iraqi Journal for Applied Science (IJAS)
Vol. 2, No. 2, June, 2025, pp. 41-48
ISSN: 3006-5828, DOI: 10.69923/b6eyv538

Model order reduction (M.O.R) of Lumped Parameter System Identification (L.P.S.I.) via Piecewise Constant Systems (P.C.S.'s)

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Article Info

Article history:

Received: 14, 01, 2025
Revised: 25, 03, 2025
Accepted: 25, 05, 2025
Published: 30, 06, 2025

Keywords:

Lumped Parameter System
Piece wise constant systems
The integration operational Matrix(I.O.M.)
The generalized integration Operational matrices (G.I.O.M.)

ABSTRACT

In this article, we give an approximation estimation parameter for model order reduction of linear time-invariant single-input single-output (S.I.S.O.) system described by the differential equation using piecewise constant systems. The operational matrices of integration, product, and generalized integration operational matrices (G.I.O.M.) for piecewise constant systems are given to convert the computation of identification of the Lumped parameter system to a simple system of algebraic equations. By using the proposed method on numerical analysis example, we show that our result has a good degree of accuracy in transmitting the transfer function of n-*th* order into γ-*th* order systems. Therefore, the proposed method is simple in implement theory and flexible in application. The first part of the article, being tutorial in nature, is on piecewise constant systems, the middle part develops an integration operational matrices method for solving order reduction of linear time-invariant single-input single-output (S.I.S.O.) system.

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1. INTRODUCTION

In recently years, several numbers of algorithms for estimating the parameters of the reduced order L.T.I. system have been investigated. Vined and Tiwari [1] used a mixed algorithm for single-input single-output stable system based on clustering technique and factor division algorithm. They showed that this method is an efficient and takes little computational time. Wavelets analysis such as shifted Legendre polynomials as a new approach of mathematics which is applied in the M.O.R. of linear time invariant and time variation systems in order to estimate the parameters in single-input single-output and multi-input multi-output (M.I.M.O.), [2,16]. Anirudha, Dinesh, and Ravindra, [3] used a numerical algorithm to stability of reduced-order model based on cluster the poles of the high-order system based on the Fuzzy C-Means clustering technique retaining some dominant poles. The bat algorithm is used to M.O.R. of large order system in [4]. Their results are satisfactory in terms of minimum error compared with Routh-Pade approximation. A new proposed method was given in [5,8] to estimate the coefficients for M.O.R. of L.I.T. system. Tiwari and Kaur [6] suggested model reduction technique for large scale stable linear dynamic systems and designed compensatory by using moment matching algorithm with the help of the M.O.R. Ali, Hussain, Mahmoud[7], discussed and designed an numerical analysis order reduction for high order sampled data systems. In their results, they obtained an efficient approximation for M.O.R.

Consider the following linear time invariant (L.T.I) single-input single-output (S.I.S.O) system which is described by the differential equation.

$$a_0 y(t) + \sum_{i=1}^{n} a_i\, y^{(i)}(t) = b_0 x(t) + \sum_{\ell=1}^{\eta} b_\ell\, x^{(\ell)}(t) \quad (1)$$

where, the coefficients: $a_0,\ a_1, a_2, \cdots, a_n\ , b_0, b_1, b_2, \cdots, b_\eta$, are known system parameters, $x^{(\ell)}(t)$ and $y^{(i)}(t)$ are the ℓ *-th* and i *-th* derivatives of the input $x(t)$ and output $y(t)$respectively with $\eta \le n$Or in terms of transfer function, the system described by eq.(1) can be written as

$$\mathbb{G}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)} = \frac{\sum_{\ell=0}^{\eta} b_\ell\, s^\ell}{\sum_{i=0}^{n} a_i\, s^i}$$

The model order reduction (M.O.R.) for the input-output differential equation of L.T.I. system will be described by:

$$\lambda_0 y_r(t) + \sum_{\theta=1}^{\gamma} \lambda_\theta\, y_r^{(\theta)}(t) = \xi_0 x(t) + \sum_{\Theta=1}^{\vartheta} \xi_\Theta\, x^{(\Theta)}(t) \quad (2)$$

where, $\vartheta \le \gamma < n$, $x(t)$ is the input and $y_r(t)$ is the output of the reduced order L.T.I. system;λ_0, $\lambda_1, \lambda_2, \cdots, \lambda_\gamma\ , \xi_0, \xi_1, \xi_2, \cdots, \xi_\vartheta$ are the parameters of the reduced order L.T.I. system is to be identified. The transfer function of eq.(2) is given via:

$$\mathbb{G}^*(s) = \frac{\mathcal{Y}^*(s)}{\mathcal{X}^*(s)} = \frac{\sum_{\Theta=0}^{\vartheta} \xi_\Theta\, s^\theta}{\sum_{\theta=0}^{\gamma} \lambda_\theta\, s^\theta}$$

In this paper, first, we designed a new method for reduced higher order of Lumped parameter system identification (L.P.S.I.) based on piece wise constant systems with their I.O.M. and G.I.O.M. Second, in estimation of M.O.R. a set of linear equations are analyzed based on the ordinary least-squares algorithm. Then, we compared our results in case of using I.O.M. and G.I.O.M. with the result in [7]. Finally, it is important to mention that, the calculation in this work is simplified MATLAB 2020a computer software.

2. METHODOLOGY (PIECE WISE CONSTANT SYSTEMS (P.C.S's))

Piece wise constant systems, are mathematical systems that are constant within specific intervals, or pieces and change value abruptly at the boundaries between these intervals. In this section, we will interest in well-known constant systems: block-pulse, Walsh, Haar and Rationalized Haar systems.

Definition (2.1): " Block-Pulse System (B.P.S)": The block-pulse system $\{\mathfrak{B}_j(t)\}_{j=0}^{\infty}$ composed of step functions, defined in [0,1), as, [9]:

$$\mathfrak{B}_i(t) = \begin{cases} 1 & t \in [p_1, p_2) \\ 0 & o.w. \end{cases} \quad (3)$$

where, $p_1 = \frac{i}{k}\ , p_2 = \frac{i+1}{k}, i = 0,1, \cdots, k-1, k = 2^r$ for some positive integer $r \in \mathcal{N}$($\mathcal{N}$ is natural numbers), are known as block-pulse functions.

B.P.S is complete orthonormal in Hilbert space $\mathcal{L}^2[0,1)$. They are applied in various areas of mathematics, particularly signal processing and approximation theory. B.P.S. is commonly used to represent and analyze signals with localized features or impulses in time or space [10].

Definition (2.2): " Walsh System $\{\mathcal{W}_j(t)\}_{j=0}^{\infty}$(W.S.)": Let $\ell \in \mathcal{N} \cup \{0\}$, is natural numbers, the binary representation of each ℓ can be defined as:

$$\ell = \sum_{n=0}^{\infty} 2^n\, \ell_n$$

where, $\ell_n \in \mathbb{F}^2 = \{0,1\}$, ($\mathbb{F}^2$ is called Galois field), then the set of Walsh functions are defined by :

$$\mathcal{W}_0(t) \equiv 1, \forall t \in [0,1) \quad (4)$$

$$\mathcal{W}_\ell(t) = \prod_{j=0}^{\infty} (\Re_j(t))^{\ell_j} \quad (5)$$

where, $\Re_j(t)$ refers to the Rademacher functions which are defined by:

$$\Re_j(t) = sign(\sin(2^j \pi t)) \quad \forall\, t \in [0,1) \quad (6)$$

where,$j = 0,1, \ldots, Log_2(k), k = 2^r$, for some $r \in \mathcal{N}$, where j is the order of Rademacher function and

$$sign(t) = \begin{cases} +1 & if \quad t > 0 \\ -1 & if \quad t < 0 \\ 0 & if \quad t = 0 \end{cases} \quad (7)$$

Walsh functions are formed periodic, orthonormal, and complete over the interval. [0,1) and was discovered by Walsh J.L. in 1923, [11], while the Rademacher functions were first introduced by Germany mathematician Hans Adolph Rademacher in 1922.

Definition (2.3): " Haar System $\{\mathcal{H}_j(t)\}_{j=0}^{\infty}$(H.S.)": The Haar system $\{\mathcal{H}_j(t)\}_{j=0}^{\infty}$ composed of step functions, defined in the semi-open interval [0,1), as, [12]:

The first step function

$$\mathcal{H}_0(t) = \frac{1}{\sqrt{n}}\begin{Bmatrix} 1 & 0 \le t < 1 \\ 0 & o.w. \end{Bmatrix} \tag{8}$$

The second step function

$$\mathcal{H}_1(t) = \frac{1}{\sqrt{n}}\begin{Bmatrix} 1 & 0 \le t < 1/2 \\ -1 & 1/2 \le t < 1 \\ 0 & o.w. \end{Bmatrix} \tag{9}$$

In general,

$$\mathcal{H}_j(t) = \frac{1}{\sqrt{n}}\begin{Bmatrix} 2^{w/2} & h/2^{w} \le t < (h+\frac{1}{2})/2^{w} \\ -2^{w/2} & (h+\frac{1}{2})/2^{w} \le t < (h+1)/2^{w} \\ 0 & o.w. \end{Bmatrix} \tag{10}$$

where, $j = 1,2,\cdots,n-1$ is the series index number and $n = 2^{r}$ is a positive integer. An w and r denote the integer decomposition of the index j, i.e. $j = 2^{w} + h$ in which $w = 0,1,2,\cdots,r-1$ and $h = 0,1,2,\cdots,2^{w} - 1$.

This set is orthogonal, orthonormal, periodic and complete functions over the semi-interval [0,1), [12].

Definition (2.4): " Rationalized Haar System$\{\mathcal{R}_j(t)\}_{j=0}^{\infty}$(R.H.S.)": The rationalized Haar system $\{\mathcal{R}_j(t)\}_{j=0}^{\infty}$ considered by Ohkita and Kobayashi [13]:

$$\begin{Bmatrix} \mathcal{R}_0(t) \equiv 1 \;\forall\, t: 0 \le t < 1 \\ \mathcal{R}_j(t) = \begin{Bmatrix} 1 & z_1 \le t < z_{1/2} \\ -1 & z_{1/2} \le t < z_0 \\ 0 & o.w. \end{Bmatrix} \end{Bmatrix} \tag{11}$$

where, $z_\rho = \frac{\mu-\rho}{2^{i}}$, $\rho = 0, \frac{1}{2}, 1$ and $j = 2^{i} + \mu - 1, i = 0,1,2,\ldots, \mu = 1,2,\ldots,2^{i}$.

R.H.S. is belong to the class of orthogonal, orthonormal, periodic and complete functions over the semi-interval [0,1), [13-14].

2.1. STATE MODEL OF FUNCTIONS APPROXIMATION IN TERMS OF P.C.S's.

Let $\mathcal{F}(t)$ be an arbitrary square integral function in the interval [0,1), then it can be approximated in an infinite series of piece wise constant step functions $\mathcal{X}_m(t)$(The block-pulse, or Walsh or Haar or Rationalized Haar functions)

$$\mathcal{F}(t) = \sum_{m=0}^{\infty} \mathcal{F}_m \mathcal{X}_m(t) \tag{12}$$

where,

$$\mathcal{F}_m = \theta_m \int_0^1 \mathcal{F}(t)\mathcal{X}_m(t)dt \tag{13}$$

where, θ_m is the normalization factor of the orthogonal functions.

Table 1 shows the normalization factor of the orthogonal functions for block-pulse, Walsh, Haar, and Rationalized Haar systems, respectively.

Table 1. The normalization factor of the P. C. S.

Piecewise Constant Systems	The normalization factor of the orthogonal functions $\boldsymbol{\theta_m}$.
Block-pulse system	m
Walsh system	1
Haar system	1
Rationalized Haar system	2^{v} , v is a positive integer

If the function $\mathcal{F}(t)$ is approximated as piece wise constant in each subinterval, then eq.(12) will be terminated after $\varkappa$ −terms, hence $\mathcal{F}(t)$can be written in the form as:

$$\mathcal{F}(t) \approx \sum_{m=0}^{\delta-1} \mathcal{F}_m \mathcal{X}_m(t) \tag{14}$$

Eq.(14) can be written in discrete form

$$\boldsymbol{\mathcal{F}(t) = \mathcal{F}_\delta^{T} \mathcal{X}_\delta(t)} \tag{15}$$

where,

$$\boldsymbol{\mathcal{F}_\delta^{T}} = [\mathcal{F}_0\, \mathcal{F}_1 \cdots \mathcal{F}_{\delta-1}] = \boldsymbol{\mathcal{F}(t)[\mathcal{X}_\delta(t)]^{T}} \tag{16}$$

is called the coefficients vector, $\boldsymbol{\mathcal{X}_\delta(t)} = [\mathcal{X}_0(t)\; \mathcal{X}_1(t) \cdots \mathcal{X}_{\delta-1}(t)]^{T}$ is piecewise constant function vector, and the symbol T is refers to the transpose and the symbol δ refers to the dimension of vectors and matrices.

Table 2 shows the coefficient vector $\mathcal{F}_{\delta}^{T}$ for P.C.S.'s. Since piece wise constant systems are the belong to a class of complete orthonormal systems in $\mathcal{L}^2[0,1)$, they can be assembled as a square matrices of order δ by dividing the closed interval [0,1] into δ – sub intervals with the length $\frac{1}{\delta}$, where $\delta = 2^r$. We denote the collection points by: $t_c = \frac{2c-1}{2\delta}$, where, $c = 1,2,\cdots,\delta$ and

Table 2. The computing of $\mathcal{F}_{\delta}^{T}$.

Piece wise Constant Systems	$\mathcal{F}_{\delta}^{T}$
Block-pulse system	$\mathcal{F}(t)[\mathfrak{B}_{\delta}(t)]^T$
Walsh system	$\mathcal{W}_{\delta}(t)\mathcal{F}(t).\frac{1}{m}$
Haar system	$\mathcal{F}(t)[\mathcal{H}_{\delta}(t)]^T$
Rationalized Haar system	$2^{r}\mathcal{R}_{\delta}(t)\mathcal{F}(t).\frac{1}{m}$

$$\mathcal{X}(j,t_c) = [\mathcal{X}\left(j,\tfrac{1}{2\delta}\right) \quad \mathcal{X}\left(j,\tfrac{3}{2\delta}\right) \quad . \; . \; . \quad \mathcal{X}\left(j,\tfrac{2c-1}{2\delta}\right) \quad . \; . \; . \quad \mathcal{X}\left(j,\tfrac{2c-1}{2\delta}\right)] \tag{17}$$

where, $j = 0,1,2, \; . \; . \; ., \delta - 1$, then the piece wise constant matrix of order δ in eq.(15) will be rewrite by:

$$[\mathcal{X}]_k = \begin{bmatrix} \mathcal{X}\left(0,\frac{1}{2\delta}\right) & \mathcal{X}\left(0,\frac{1}{2\delta}\right) & \cdots & \mathcal{X}\left(0,\frac{1}{2\delta}\right) & \cdots & \mathcal{X}\left(0,\frac{1}{2\delta}\right) \\ \mathcal{X}\left(1,\frac{1}{2\delta}\right) & \mathcal{X}\left(1,\frac{1}{2\delta}\right) & \cdots & \mathcal{X}\left(1,\frac{1}{2\delta}\right) & \cdots & \mathcal{X}\left(1,\frac{1}{2\delta}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{X}\left(\delta-1,\frac{1}{2\delta}\right) & \mathcal{X}\left(\delta-1,\frac{1}{2\delta}\right) & \dots & \mathcal{X}\left(\delta-1,\frac{1}{2\delta}\right) & \dots & \mathcal{X}\left(\delta-1,\frac{1}{2\delta}\right) \end{bmatrix} \tag{18}$$

2.2. INTEGRATION OPERATIONAL MATRIX OF P.C.S's

Definition(2.2.1): Let $\boldsymbol{\mathcal{X}_{\delta}(t)} = [\mathcal{X}_0(t) \; \mathcal{X}_1(t) \; \cdots \; \mathcal{X}_{\delta-1}(t)]^T$ be a basis piece-wise constant function vector, then the integration operational matrix (I.O.M.) of $\boldsymbol{\mathcal{X}_{\delta}(t)}$ is square matrix obtained by integrating each element in $\boldsymbol{\mathcal{X}_{\delta}(t)}$, and expressing mathematically by the following formula:

$$\int_0^t \boldsymbol{\mathcal{X}_{\delta}(\sigma)} d\sigma = \mathcal{Q} \, \boldsymbol{\mathcal{X}_{\delta}(t)} \tag{19}$$

where, $\mathcal{X}_0(t) \; \mathcal{X}_1(t) \; \cdots \; \mathcal{X}_{\delta-1}(t)$ are basis piece wise constant step functions and $\mathcal{Q}$ is the integration operational matrix of $\boldsymbol{\mathcal{X}_{\delta}(t)}$.

B.P.S.

$$\mathcal{Q} = \frac{1}{\delta}\begin{pmatrix} 1/2 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1/2 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1/2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1/2 \end{pmatrix}$$

W.S.

$$\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_{\left(\frac{n}{2}\times\frac{n}{2}\right)} & -\frac{1}{2n}[I]_{\left(\frac{n}{2}\times\frac{n}{2}\right)} \\ \frac{1}{2n}[I]_{\left(\frac{n}{2}\times\frac{n}{2}\right)} & \boldsymbol{O}_{\left(\frac{n}{2}\times\frac{n}{2}\right)} \end{bmatrix}$$

where, $\mathcal{Q}_{(1\times1)} = \left[\frac{1}{2}\right]$.

H.S.

$$\mathcal{Q} = \frac{1}{2n}\begin{bmatrix} 2n\mathcal{Q}_{\frac{n}{2}} & -[\mathcal{H}]_{\frac{n}{2}} \\ ([\mathcal{H}]_{\frac{n}{2}})^{-1} & \boldsymbol{O} \end{bmatrix}$$

where, $\mathcal{H}$ is Haar matrix of order $\frac{n}{2}$, and $\boldsymbol{O}$ is a null matrix of order $\frac{n}{2} \times \frac{n}{2}$.

R.H.S.

$$\mathcal{Q} = \frac{1}{2n}\begin{bmatrix} 2n\mathcal{Q}_{\frac{n}{2}} & -[\mathcal{R}]_{\frac{n}{2}} \\ ([\mathcal{R}]_{\frac{n}{2}})^{-1} & \boldsymbol{O} \end{bmatrix}$$

where, $[\mathcal{R}]_1 = [1], \mathcal{Q}_1 = \left[\frac{1}{2}\right]$, and $\boldsymbol{O}$ is a null matrix of order $\frac{n}{2} \times \frac{n}{2}$.

For $\hbar$ −times repeated integration of piecewise constant function vector, we get:

$$\underbrace{\int_0^t\int_0^t\int_0^t\cdots\int_0^t \mathcal{X}_j(x)dx^{\hbar}}_{\hbar-times} \sim Q^{\hbar}\boldsymbol{\mathcal{X}}_{\boldsymbol{n}}(t) \quad \forall\, j=0,1,\cdots,n-1. \tag{20}$$

where, $Q_{\hbar}$ is known as the G.I.O.M. of $\boldsymbol{\mathcal{X}}_{\boldsymbol{\delta}}(t)$ in terms of B.P.F's.

3. RESULTS AND DISCUSSION

In this section, we will be modify the integration operational matrix of P.C.S's. in section (2.2) based on the term generalized integration operational matrices (G.I.O.M.). This term was found in applying the block pulse system technique to the problems of continuous-time dynamic systems, [15,17]. We will extend this term to include Walsh, Haar, and rationalized Haar functions in terms of block pulse functions.

Definition(3.1): Let $\boldsymbol{\mathcal{X}}_{\boldsymbol{\delta}}(t)=[\mathcal{X}_0(t)\ \mathcal{X}_1(t)\ \cdots\ \mathcal{X}_{\delta-1}(t)]^T$ be a basis piece wise constant function vector, then the $\hbar$ times integrals of all $j$piece wise constant function vector can be written together in a compact matrix form:

$$\underbrace{\int_0^t\int_0^t\int_0^t\cdots\int_0^t \boldsymbol{\mathcal{X}}_{\boldsymbol{\delta}}(x)dx^{\hbar}}_{\hbar-times} \sim \mathcal{Q}_{\hbar}\boldsymbol{\mathcal{X}}_{\boldsymbol{\delta}}(t) \tag{21}$$

where, $\mathcal{Q}_{\hbar}$is called the $\hbar$ -th generalized integration operational matrix.

For example the $\hbar$ -th generalized integration operational matrix for block-pulse functions is given by,[15]:

$$\mathcal{Q}_{\hbar}^{(\boldsymbol{Bb})}=\frac{1}{m^{\hbar}(\hbar+1)!}\begin{bmatrix} \mathcal{G}_{\hbar,1} & \mathcal{G}_{\hbar,2} & \mathcal{G}_{\hbar,3} & \cdots & \mathcal{G}_{\hbar,m} \\ 0 & \mathcal{G}_{\hbar,1} & \mathcal{G}_{\hbar,2} & \cdots & \mathcal{G}_{\hbar,m-1} \\ 0 & 0 & \mathcal{G}_{\hbar,1} & \cdots & \mathcal{G}_{\hbar,m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{G}_{\hbar,1} \end{bmatrix}$$

with

$$\mathcal{G}_{\hbar,j}=\begin{bmatrix} 1 & for\ j=1 \\ j^{\hbar+1}-2(j-1)^{\hbar+1}+(j-2)^{\hbar+1}\cdots,m & for\ j=2,3, \end{bmatrix}$$

Since the orthogonal functions: Walsh, Haar and rationalized Haar functions are belong to square-integrable functions in the semi-open interval [0,1), then they can be approximated by a δ −set of complete orthonormal block-pulse functions as follow:

$$\boldsymbol{\mathcal{X}}_i(t)=\sum_{j=0}^{\delta-1}\xi_{ij}\,\mathfrak{B}_j(t) \tag{22}$$

where, $i=0,1,2,\ \ldots,\delta-1$, $\boldsymbol{\mathcal{X}}_i(t)$ refers to the Walsh or Haar or rationalized Haar functions and ξ_{ij} denotes the coefficients of $\boldsymbol{\mathcal{X}}_i(t)$ which their values are the Walsh or Haar or Rationalized Haar matrices respectively. Eq.(22) can be written in a square matrix form of order δ:

$$[\boldsymbol{\mathcal{X}}]_{\delta}=[\xi]_{\delta}[\mathfrak{B}]_{\delta} \tag{23}$$

$$\begin{bmatrix} \mathcal{X}_0(t) \\ \mathcal{X}_1(t) \\ \vdots \\ \mathcal{X}_{\delta-1}(t) \end{bmatrix}=\begin{bmatrix} \xi_{00} & \xi_{01} & \xi_{02} & \cdots & \xi_{0k-1} \\ \xi_{10} & \xi_{11} & \xi_{12} & \cdots & \xi_{1k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{(\delta-1)0} & \xi_{(\delta-1)1} & \xi_{(\delta-1)2} & \cdots & \xi_{(\delta-1)(\delta-1)} \end{bmatrix}\begin{bmatrix} \mathfrak{B}_0(t) \\ \mathfrak{B}_1(t) \\ \vdots \\ \mathfrak{B}_{\delta-1}(t) \end{bmatrix}$$

By substituting eq.(23) into eq.(21), we have

$$\int_0^t\int_0^t\int_0^t\cdots\int_0^t[\boldsymbol{\xi}]_{\delta}[\boldsymbol{\mathfrak{B}}]_{\delta}dx^{\hbar}=\mathcal{Q}_{\hbar}[\xi]_{\delta}[\mathfrak{B}]_{\delta}$$

$$[\boldsymbol{\xi}]_{\delta}\mathcal{Q}_{\hbar}^{(\boldsymbol{BP})}[\boldsymbol{\mathfrak{B}}]_{\delta}=\mathcal{Q}_{\hbar}[\boldsymbol{\xi}]_{\delta}[\boldsymbol{\mathfrak{B}}]_{\boldsymbol{\delta}}$$

$$\mathcal{Q}_{\hbar}=[\boldsymbol{\xi}]_{\delta}\boldsymbol{\mathcal{Q}}_{\hbar}^{(\boldsymbol{BP})}([\boldsymbol{\xi}]_{\delta})^{-1} \tag{24}$$

For a single-input single-output time-invariant linear system, we integrate the differential equation eq.(1) n- times from 0 to t on both sides under zero initial values, we obtain

$$y(t)+a_{n-1}\int_0^t y(t)+\cdots+a_0\underbrace{\int_0^t\cdots\int_0^t y(t)\mathrm{d}t\cdots\mathrm{d}t}_{\mathrm{n-times}}$$

$$=b_0\underbrace{\int_0^t\cdots\int_0^t x(t)\mathrm{d}t\cdots\mathrm{d}t}_{\mathrm{n-times}}+b_1\underbrace{\int_0^t\cdots\int_0^t x(t)\mathrm{d}t^{\mathrm{n}-1}}_{\mathrm{n-1-times}}\ +b_{\eta}\underbrace{\int_0^t\cdots\int_0^t x(t)\mathrm{d}t^{\mathrm{n}-\eta}}_{\mathrm{n-\eta-times}} \tag{25}$$

The input and output signals are now expanded in a finite series of piece wise constant systems as:

$$x(t)\approx\sum_{m=0}^{\delta-1}x_m\,\mathcal{X}_m(t)=\boldsymbol{x}_{\boldsymbol{\delta}}^{\boldsymbol{T}}\boldsymbol{\mathcal{X}}_{\boldsymbol{\delta}}(\boldsymbol{t}) \tag{26}$$

$$y(t)\approx\sum_{m=0}^{\delta-1}y_m\,\mathcal{X}_m(t)=\boldsymbol{y}_{\boldsymbol{\delta}}^{\boldsymbol{T}}\boldsymbol{\mathcal{X}}_{\boldsymbol{\delta}}(\boldsymbol{t})$$

Then, the generalized integration operational matrices (or I.O.M.) method in eq. (24), we obtain the piece wise constant series of the eq.(26) in a vector form:

$$\boldsymbol{y}_\delta^T(\mathbb{I}+a_{n-1}\mathcal{Q}_1+\cdots+a_1\mathcal{Q}_{n-1}+a_0\mathcal{Q}_n)\boldsymbol{X}_\delta(\boldsymbol{t})$$
$$=\boldsymbol{x}_\delta^T(b_\eta\mathcal{Q}_{n-\eta}+b_{n-1}\mathcal{Q}_1+\cdots+b_1\mathcal{Q}_{n-1}+b_0\mathcal{Q}_n)\boldsymbol{X}_\delta(\boldsymbol{t})$$

Equating the coefficients of each piecewise constant function separately on both sides of the above equation, we have:

$$\boldsymbol{y}_\delta^T\mathcal{A}=\boldsymbol{x}_\delta^T\mathcal{B} \quad (27)$$

where,

$$\mathcal{A}=\mathbb{I}+a_{n-1}\mathcal{Q}_1+\cdots+a_1\mathcal{Q}_{n-1}+a_0\mathcal{Q}_n$$

and

$$\mathcal{B}=b_\eta\mathcal{Q}_{n-\eta}+b_{n-1}\mathcal{Q}_1+\cdots+b_1\mathcal{Q}_{n-1}+b_0\mathcal{Q}_n$$

If $\mathcal{A}$ is nonsingular, that of the input can determine the piece wise constant coefficient vector of the output:

$$\boldsymbol{y}_\delta^T=\boldsymbol{x}_\delta^T\mathcal{B}\mathcal{A}^{-1} \quad (28)$$

where $\mathcal{A}$ and $\mathcal{B}$ are constant matrices depending on the known parameters of the original system. Using the same procedure as described above, after integrating the differential equation in eq.(2) n- times from 0 to t on both sides under zero initial values, we obtain the form of the reduced order system by P.C.S's:

$$\boldsymbol{y}_{\delta r}^T=\boldsymbol{x}_\delta^T\boldsymbol{y}_\delta^T=\boldsymbol{x}_\delta^T\mathcal{B}_r\mathcal{A}_r^{-1} \quad (29)$$

where,

$$\mathcal{A}_r=\mathbb{I}+\lambda_{\theta-1}\mathcal{Q}_1+\cdots+\lambda_1\mathcal{Q}_{\gamma-1}+\lambda_0\mathcal{Q}_\gamma$$

And

$$\mathcal{B}_r=\xi_\eta\mathcal{Q}_{\gamma-\vartheta}+\xi_{\gamma-1}\mathcal{Q}_1+\cdots+\xi_1\mathcal{Q}_{\gamma-1}+\xi_0\mathcal{Q}_\gamma$$

are matrices depending on the parameters of the reduced order system and $\mathcal{Q}_d$is the generalized integration operational matrices of integration depending of the chosen P.C.S's.

Since M.O.R. has the property that it has a similar input $x(t)$ and output $y(t)$-output to the original system in eq.(1) for all inputs $x(t)$, then we have the following relation:

$$\boldsymbol{y}_\delta^T\boldsymbol{X}_\delta(\boldsymbol{t})=\boldsymbol{y}_{\delta r}^T\boldsymbol{X}_\delta(\boldsymbol{t})\Leftrightarrow\boldsymbol{y}_\delta^T=\boldsymbol{y}_{\delta r}^T$$

to obtain the ordinary least-squares algorithm

$$\mathcal{T}\mathcal{O}=\boldsymbol{y}_{\delta r}^T$$

where,

$$\mathcal{T}=[-\mathcal{Q}_1^{\mathrm{T}}\boldsymbol{y}_{\delta r}|-\mathcal{Q}_2^{\mathrm{T}}\boldsymbol{y}_{\delta r}|\cdots|-\mathcal{Q}_{\gamma-1}^{\mathrm{T}}\boldsymbol{y}_{\delta r}|-\mathcal{Q}_\gamma^{\mathrm{T}}\boldsymbol{y}_{\delta r}|\mathcal{Q}_{\gamma-\vartheta}^{\mathrm{T}}\boldsymbol{x}_\delta|\mathcal{Q}_{\gamma-\vartheta+1}^{\mathrm{T}}\boldsymbol{x}_\delta|\cdots|\mathcal{Q}_{\gamma-1}^{\mathrm{T}}\boldsymbol{x}_\delta|\mathcal{Q}_\gamma^{\mathrm{T}}\boldsymbol{x}_\delta]$$
$$\mathcal{O}=[\lambda_{\gamma-1},\lambda_{\gamma-2},\cdots,\lambda_1,\lambda_0,\lambda_2,\cdots,\xi_\vartheta,\xi_{\vartheta-1},\xi_1,\xi_0]$$

and the least squares estimate of the $\mathcal{O}$ is given by

$$\mathcal{O}^*=[\mathcal{T}^T\mathcal{T}]^{-1}[\mathcal{T}^T\boldsymbol{y}_{\delta r}]$$

This relation makes it possible to estimate all the parameters of the model order reduction (M.O.R.) system.

Example(3.1): Consider a system described via the fourth-order differential equation

$$\mathbb{G}(s)=\frac{0.07844s^3-0.1556s^2+0.1042s-0.02388}{s^4-2.698s^3+2.643s^2-1.106s+0.1653}$$

We want to estimate the parameters. $p_0, p_1, p_2, q_1, and\ q_2$ for the second-order system of the following form

$$\mathbb{G}^*(s^{-1})=\frac{p_0+p_1s^{-1}+p_2s^{-2}}{1+q_1s^{-1}+q_2s^{-2}}$$

The estimated values of the parameters for the transfer function $\mathbb{G}^*(s^{-1})$, with, $m=8$ derived via four types of **P.C.S's** i.e. **B.P.S.**, **W.S. H.S.**, and **R.H.S.** are shown in Table 3.

Table 3. Comparative study of the parameters of the second order system under investigation in different **P.C.S's** domains.

	Parameters	p_0	p_1	p_2	q_1	q_2
	Actual parameters	0	0.0784	-0.0297	-1.0929	0.1588
	Estimation parameters in [7]	0	0.0448	-0.0373	-1.822	0.8327
Block Pulse System	G.I.O.M.	0	0.0784	-0.0412	-1.2398	0.2725
	I.O.M	0	0.0787	-0.0426	-1.2616	0.2829
Walsh System	G.I.O.M.	0	0.0784	-0.0412	-1.2398	0.2725
	I.O.M	0	0.0787	-0.0426	-1.2616	0.2829
Haar System	G.I.O.M.	0	0.0784	-0.0412	-1.2398	0.2725
	I.O.M	0	0.0787	-0.0426	-1.2616	0.2829
Rationalized Haar System	G.I.O.M.	0	0.0784	-0.0412	-1.2393	0.2722
	I.O.M	0	0.0787	-0.0426	-1.2612	0.2826

The comparative study of the error estimates for the system under study for $m = 8$ via block pulse system, Walsh system, Haar system, rationalized Haar system, are shown in Table 4**.**

Table 4. Comparative study of error estimates of the parameters for the second order system under investigation in different P.C.S's domains.

	Parameters	p_0	p_1	p_2	q_1	q_2
	Exact parameters	0	0.0784	-0.0297	-1.0929	0.1588
	Error analysis in [7]	0	0.0336	0.0076	0.7291	0.6739
Error analysis in Pricewise constant Systems Error=|exact parameters-Θ^*|	G.I.O.M.	0	0.0000	0.0115	0.1469	0.1137
	I.O.M.	0	0.0003	0.0129	0.1687	0.1241

4. CONCLUSION

In view of the result for the second order of the distributed parameter system in example (3.1) using all classes of orthogonal systems, it appears that the proposed method using Piecewise constant systems is the simplest as it is very fast and also computationally very attractive. Piecewise constant systems study this example via considering $m = 8$, and the input and output signals over the region $t \in [0,1]$ to estimate the parameters. It can be noted that the estimates obtained by G.I.O.M. and I.O.M. are better than that obtained by the proposed method in [7]. In this case, the estimates can be further improved by considering the following factors: $m = 16,32,\ldots,$ in the series expansion. The identification algorithm developed in section (3) can be extended to include any type of orthogonal systems (polynomials: Shifted Tchebycheff polynomials of first and second types, shifted Legendre polynomials, and other piecewise constant systems such as Sample and Hold functions(S.H.F.)).

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